Quantum Field Theories at Small Distances*

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The behaviour of the propagation functions of quantized field theories at small distances is investigated in connection with the problem of consistency of the perturbation theoretical renormalization scheme. An attempt is made to adapt the conventional Hamiltonian and S-matrix formalism to the renormalization concept in such a way that a finite theory of interacting physical (dressed) particles results which as a whole and at each step of approximation satisfies the axioms of the general structure theory of quantized fields, notably causality and positive definiteness. If the physical particles of a field theory with point interaction are extended objects owing to the extension of the cloud of virtual quanta in the physical states, then the theory admits of a finite formulation with the extent of the cloud introducing a natural, coupling-dependent built-in cutoff. The limiting case of a point-like cloud corresponds to physical particles which because of their strong self-interaction do not interact with one another. This case, corresponding to the zero coupling limit, represents the most singular one given by the theory. An increase of interaction or coupling implies an increase of the size of the cloud and a corresponding charge spread. Conversely, there cannot be interaction if there is no extended cloud and charge spread. Cloud and charge structure come in via causal form factors related to vertex parts and electromagnetic form factors, which however a perturbation theoretical scheme cannot take into account.

1. Introduction and Discussion

1.1. Phenomenological Considerations

If the physical particles of a quantum field theory with point interaction are extended objects owing to the extension of the cloud of virtual quanta in the physical states. then the theory should be finite if it is formulated in such a way that the size of the cloud introduces a natural built-in cutoff. The theory then also should be convergent in an approximative description which at each step accounts explicitly and in a mathematically consistent way for the extended cloud structure of the particles. The conventional perturbation approach to quantized field theories does not have this property: Since the perturbation approximations to the vertex function do not vanish sufficiently rapidly for large momenta. the dominating contribution of virtual quanta with arbitrarily high momenta causes the cloud of the physical particles to be point-like (see sections 3 and 4). This manifests itself in a divergent [1] wave function normalization factor Z^{-1} and implies a mapping of the approximate theory into a free one by renormalization if one insists that the theory have the correct axiomatic structure. The reduction of the interaction to zero sometimes is reinterpreted in terms of ghost states by reintroducing interaction at the expense of destroying the axiomatic structure of the theory.

At first sight, the observable corrections to the Coulomb potential due to vacuum polarization in quantum electrodynamics would seem to contradict the statement that perturbation approximations only contribute a point-like cloud to the physical particles since those corrections result already from the formally renormalized, non-iterated electron-positron bubble. However. within the frame of an axiomatic local theory the bubble approximation of the photon propagator (and any finite-order iteration of it) cannot be renormalized in a consistent and not only formal way unless both the renormalized and the unrenormaziled coupling constants. g and g_0 respectively, vanish identically. For a nonzero g, a consistent renormalization within the considered order of approximation requires the introduction of a high-energy cutoff $1/\lambda$ such that the corresponding approximate Z-factor $[Z \simeq 1 + g^2 \log(m\,\lambda)]$

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[1] The wave function renormalization Z^{-1} could diverge also in case of a decreasing vertex function if the unrenormalized coupling would turn out to be infinite. This would not necessarily imply a free theory. In what follows, if we refer to a divergent Z^{-1} we suppose this divergency to be due to an insufficiently rapidly decreasing vertex part or, more generally, absorptive part of the (renormalized) proper self-energy.

comes arbitrarily close to one. This presupposes of course that the exact local theory be mathematically well-defined for $g^2>0$ (for the contrary case cf. the later discussion) and that the cutoff does not make itself felt for momenta much smaller than $1/\lambda$. Hence, within the frame of a consistent renormalization formalism the observable corrections derived from the bubble approximation appear as being due to the fact that the cutoff reveals the cloud structure of the physical particle for distances much larger than λ. However, the main contributions to the renormalization effects – in particular, the magnitude of the normalization factor Z which connects g and g_0 by $g^2=Z(g^2)\,g_0{}^2$ – are determined by the cloud's structure in the region $0\leq r\ll\lambda$ which owing to the cutoff can never be attained. Any finite order iteration of the bubble approximation obviously gives rise to the same situation. It is however only the sum of all the iterations of proper diagrams that can be renormalized consistently without the necessity of Z being kept close to one. In the local limit. $\lambda\to 0$, renormalization of the totally iterated bubble approximation implies again the vanishing of the renormalized coupling, $g=0$, if the theory is to be axiomatic, but this time for *any* real (finite) unrenormalized coupling g_0 in contrast with the finite-order iteration where both g and g_0 must vanish simultaneously. This situation persists also for higher approximations.

Consequently, the conventional perturbation approach explicitly admits of the possibility that $g_0\to g_0{}'=g_0(0)\neq 0$ (and finite) for $g\to 0$. If this is so, then the local interaction HAMILTONian $H'=g_0(g)H$ does not vanish as $g\to 0$ (at least to the extent that all proper totally iterated diagrams up to any finite order are considered). This implies that in the limit $g\to 0$ the HAMILTONian H' gives rise to physical particles which in virtue of their strong self-interaction do not interact with one another and thus are characterized by vanishing renormalized charge, $g=0$ and $Z=0$, and by free propagators. This rather singular type of a free particle may be visualized as consisting of a point-like core the charge $g_0{}'\neq 0$ of which is completely compensated, so as to yield $g=0$, by that of the surrounding but (in the limit $g\to 0$) still point-like cloud due to vacuum polarization (i. e., by the charge $g-g_0(g)\to -g_0{}'$). If the physical particles of the exact local theory possess an extended cloud structure, then the size of the cloud of a particle must depend on the coupling and shrink to a point as $g\to 0$ if the exact theory is supposed to reveal the above perturbation theoretical result in the limit of vanishing coupling where only the self-interaction remains $[g_0(0)\,H\neq 0]$. This implies that the exact theory is singular at zero coupling, though finite for $g^2>0$, since owing to the assumed persistence of $g_0\neq 0$ for $g^2\to +0$, the exact Z-factor behaves as $Z^{-1}=c/g^2+o(g^2)$ as $g^2\to +0$ with $c=g_0{}^2(0)\neq 0$. Hence, in accordance with the evidence provided by perturbation theory, no coupling-constant expansion of Z^{-1} is possible although a perturbation approach may supply an asymptotic representation for the exact propagator. However, as we shall see, an asymptotic expansion can be obtained only at the expense of destroying the cloud structure of the particles and even then only if the original perturbation formalism exhibits logarithmic divergences. Nevertheless, from this point of view the physical usefulness of the formally renormalized non-iterated bubble approximation could be understood. Evidently, if the weak-coupling limit $g\to 0$ with the corresponding point cloud represents the most singular case in the theory, then the increase of g^2 will introduce interaction via convergence-producing form factors (related to the vertex function and representing the extended cloud (or charge) spread of the particles) which however a perturbation theoretical expansion cannot take into account.

1.2. Outline of the Program

In the present paper, an attempt is made to adapt the conventional HAMILTONian and S-matrix formalism to the renormalization concept in such a way that a finite theory of interacting physical (= dressed) particles results which as a whole and at each step of approximation satisfies the axioms of the general structure theory of quantized fields. The postulates of the axiomatic theory [2], notably causality and positive definiteness condition, imply that the propagators of the particles given by the theory have the correct analytic structure in the cut energy-plane. The existence of the axiomatic (and, therefore, renormalized) theory implies the existence of the unrenormalized one unless $Z\equiv 0$. The three concepts, "extended cloud (or charge) structure of the physical particles", "correct analytic structure"

[2] A. S. WIGHTMAN, in „Les Problèmes Mathématiques de la Théorie quantique des Champs", C.N.R.S., Paris 1960, p. 1.

and "mathematically consistent renormalizability" of the propagators for non-vanishing interaction are essentially equivalent. By starting from a perturbation approach – actually the only one which has proved feasible for obtaining explicit results – we first account preliminarily for the future cloud structure the particles will get by the interaction by introducing an arbitrary cutoff (λ) into the original local interaction HAMILTONian (which, however, will be needed only as a generator of selection rules). The approximated unrenormalized propagator which in virtue of the cutoff has the correct analytic structure is then adapted to the axiomatic renormalization concept for arbitrarily large coupling (section 2). To each approximation there corresponds a modified HAMILTONian that coincides with the initial one (with cutoff) for small coupling but contains an additional however well-determined form factor for large coupling. This reflects itself in a splitting of the function $g_0^2(g^2)$ into two branches (see Fig. 1 a). In the

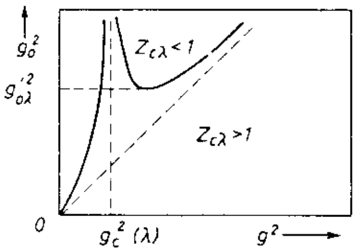

Fig. 1 a. The function $g_0^2(g^2)$ in the axiomatized cutoff theory [$\lambda > 0$, $g_c^2(\lambda) \to 0$ for $\lambda \to 0$].

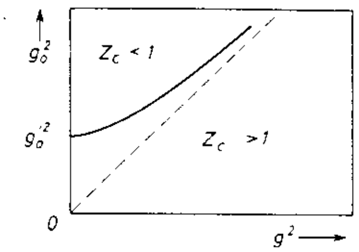

Fig. 1 b. The function $g_0^2(g^2)$ in the axiomatized theory without cutoff (limiting case of Fig. 1 a for $\lambda \to 0$, where $g_{0\lambda}'^2 \to g_0'^2$).

limit of vanishing cutoff ($\lambda \to 0$) the first branch shrinks to the point $g=0$ while the second one extends to the whole region $g^2>0$ (see Fig. 1 b). Correspondingly, the original local interaction HAMILTONian $H' = g_0 H$ defines the approximated theory only for $g=0$ while for $g^2>0$ the theory is described in terms of a modified, apparently nonlocal HAMILTONian $H'[F]$ which differs from the local one, $H' = H'[\delta]$, by a uniquely determined special form factor $F = F(x, g^2)$ which in principle can be calculated explicitly. $F(x, g^2)$ depends explicitly on the coupling, has an essential singularity at $g=0$ and $F \to \delta(x)$ asymptotically for $g^2 \to +0$. To each approximation – various possibilities exist for proceeding to higher approximations – there corresponds a new form factor, $F = F_n$, the sequence of F_n tending to $\delta(x)$ as with increasing n the approximations to the exact proper self-energy become better so that in the limit $n \to \infty$ the sequence of nonlocal HAMILTONians $H_n' = H'[F_n]$ tends to the local interaction $H' = H'[\delta]$. Since $F \to \delta$ for $g^2 \to +0$, already the first approximation (axiomatized iterated bubble approximation) may be expected to give at least a qualitative information about the small distance structure of the exact theory for sufficiently small coupling.

1.3. *Discussion of the Results*

The axiomatization of the HAMILTONian formalism furnishes the absorptive part of the approximate proper self-energy supplied by the asymptotic part H' of H'_n with a uniquely determined calculable high-energy cutoff $F_n = F_n(p, g^2)$. In virtue of these F_n the approximations to the exact spectral function of the propagator decrease sufficiently rapidly for large momenta to yield a finite wave function normalization Z^{-1} for $g^2>0$. In the axiomatic theory the primary quantity is indeed the spectral function of the propagator and not the absorptive part of the proper self-energy although in the HAMILTONian and S-matrix formalism it is just the other way around. Both quantities become now functionals of F_n. While the axiomatization formalism gives first of all those F_n, the form factors F_n in the approximate HAMILTONians $H'[F_n]$ are functionals of the F_n and must be considered as being deduced quantities which mainly serve for purposes of visualization. We do in fact not know whether a consistent superposition in $H'[F]$ of form factor contributions that originate from different types of axiomatized propagators (boson and fermion) actually is possible. A decision of this question is difficult since the axiomatic approximations necessarily are non-additive in contrast with the non-axiomatic perturbation approximations in which however the form factors do not appear at all. The sequence of finite axiomatic theories, which eventually converges to the exact local theory, has just the singular structure at zero coupling that we have guessed above (section 3). Both F_n and F_n have an essential singularity at zero coupling with $F_n \to \delta$, $F_n \to 1$ asymptotically for $g^2 \to +0$ and their expansions in powers of g^2 reduce to $\delta(x)$ and 1 respectively. Hence, the form factors do not give rise to any FEYNMAN diagram in a coupling-constant expansion of the S-matrix pertaining to $H'[F_n]$ which includes all proper totally iterated graphs up to any finite order, nor are the form factors reproducable by a perturbation theoretical approach. The formally renormalized non-axiomatic (approximate) propagators obtained from a perturbation theoretical treatment of the local theory, therefore, turn out to be asymptotic representations (for $g^2 \to +0$) of the axiomatic propagators supplied by the new approximation scheme. The propagator given by the first step of the scheme coincides with the one obtained recently by REDMOND [3] and BOGOLIUBOV et al. [4] in a formal way from a selected infinite set of graphs by summing the associated spectral functions. Our higher approximations, however, give rise to different possibilities. The introduction (into the HAMILTONian) of form factors which do not contribute to formal series expansions in powers of g^2 and the requirement of asymptotic coincidence of perturbation theoretical and axiomatized propagators can indeed be taken as a starting point for putting BOGOLIUBOV's purely formal method on a mathematically rigorous and physically meaningful basis [5]. REDMOND's techniques evidently rests upon the assumption that the exact local theory exist. This assumption needs not to be made in the theory proposed in this paper: Our formalisms still works even if the local theory does not exist, in which case the formalism automatically amounts to a modification of the theory according to the underlying principle of physical particles which possess an extended cloud structure and a probabilistic charge spread in consequence of the interaction. This new theory will then have the formally renormalized but axiomatically inconsistent local theory as an asymptotic representation (for arbitrarily weak coupling) in case the latter exhibits logarithmic divergences. This applies in particular to the LEE model [6] where the local interaction defines an axiomatic theory only for $g \equiv 0$. Evidently, even if an exact local theory would turn out to define a consistent scheme but would lose its physical meaning at small distances, the suggested approximation scheme might prove useful in a modification of the theory that accounts for the structure of the physical particles and eventually introduces new degrees of freedom but still preserves the selection rules and graph schemes which govern the original local theory.

The original local interaction HAMILTONian H' acts first of all as a generator of selection rules and of graph schemes (via the formal S-matrix expansion) but it does not necessarily determine the space-time structure and the interaction of the physical particles if the theory is to be axiomatic, i. e., if one requires the physical particles to be dressed and to interact according to the same rules (cf., e. g. the LEE model). To generate graph schemes H' is needed only in the limit of vanishingly small g^2 and taking this as a correspondence principle the theory may be determined for larger values of g^2 by imposing the general structure axioms. However, for $g^2 > 0$ it is only due to the presence in $H'[F] = g_0 F * H$ [7] of a form factor $F \neq \delta$ that the local HAMILTONian $H' = H'[\delta] = g_0 H$ becomes efficient in a power-series representation of the S-matrix since the effect of H' alone would only consist in furnishing the particles with a point-like cloud corresponding to $Z^{-1} = \infty$ and to free dressed particles. In virtue of the essential singularity at zero coupling, the form factor F in the approximate HAMILTONian $H'[F]$ does of course not contribute to the qualitative propagation scheme supplied by the asymptotic part H' of $H'[F] = F * H'$, and F and F should be interpreted as being a measure for the extent and the internal structure of the cloud of the interacting physical particles. It should indeed not be considered surprising that an axiomatized approximation scheme approaches the exact local theory in terms of a sequence of pseudononlocal [8] HAMILTONians: If the interacting physical particles of a local theory

[3] P. J. REDMOND, Phys. Rev. **112**, 1404 [1958] and preprints. — P. J. REDMOND and J. L. URETSKY, Nucl. Phys. **12**, 485 [1959]. It is one of the purposes of this paper to give a physical interpretation to REDMOND's work. The authors are grateful to Dr. REDMOND for communications in advance of publication.

[4] N. N. BOGOLIUBOV, A. A. LOGUNOV, and D. V. SHIRKOV, preprint, Dubna 1959.

[5] J. A. SWIECA, Nucl. Phys., to be published. This paper will be referred to as II.

[6] T. D. LEE, Phys. Rev. **95**, 1329 [1954].

[7] The symbol $*$ means the convolution product in coordinate space.

[8] That is to say, $S\{H'\} = S\{H'[F]\}$ in an expansion of the S-matrix in powers of g^2.

with point-interaction possess an extended cloud or charge structure via the internal form factors defined by the exact (renormalized) vertex function, then the interaction between the particles must be expected to become a visibly nonlocal one (in the sense of a nonlocal HAMILTONian only!) if an approximation to the exact vertex function is made which although depriving it of (part of) its internal form factors still maintains the extended cloud structure of the particles. If, in particular, one starts from a perturbation approach and requires the formal S-matrix expansion pertaining to H' to coincide with that one pertaining to $H'[F]$, then the form factor F into which (part of) the effects of the exact vertex are reintroduced obviously must have an essential singularity at zero coupling. From the general structure of the absorptive part of the proper self-energy it can be seen that the $F_n(p, g^2)$ play in fact the role of a vertex part or that of a contribution to the exact vertex. That is to say, if the sequence of approximate theories converges to the exact local theory – the question of convergence being of course as open as that of the existence of the exact theory – then the effects of the form factors F_n, F_n will be transferred into the exact vertex function or into the internal form factors defined by the latter. It should be clear, therefore, that if the exact local theory exists, these dynamical form factors have as little to do with the kinematical form factors of non-local field theories as the vertex part of local theories has. $F_n(p, g^2)$ is by construction a causal form factor in the sense that the vacuum expectation value of the commutator (defined via the axiomatized propagator) vanishes at space-like distances.

If one attempts to axiomatize a HAMILTONian formalism one obviously is faced with the problem of reinterpreting – rather than eliminating – the concept of bare particles. Since axiomatization means essentially consistent renormalization, this problem becomes relevant only if $Z=0$ in which case the passage from renormalized to unrenormalized quantities is not possible. What can be done at most if one insists on dealing with physical particles only (without referring to the problematic nature of field operators and bare particle states) is to compare cloud or charge structures for different values of the coupling. The fact that the structure function F (and similarly F, F_n and F_n) gives a measure for the extended charge structure of the physical particle and the statistical nature of the form factors is borne out by a projection of "bare with respect to physical vacuum" – states onto physical particle states in which the form factors play the role of weight functions (section 4). It follows quite generally from this projection that a finite normalization factor Z^{-1}, i. e., a sufficiently rapidly decreasing vertex function[1], leads to physical particles which have an extended cloud or charge spread whereas an infinite Z^{-1}, i. e., an insufficiently rapidly decreasing vertex function, renders the cloud of the physical particles to be point-like. In the axiomatized theory, the form factors F and F (resp., F_n and F_n) give rise to a finite wave function normalization $Z^{-1}(g^2)$ for $g^2>0$ and, therefore, to a cloud of finite extent which shrinks to a point (the one defined by the bare quantuum) in the limit $g^2 \rightarrow +0$ in which $Z^{-1} \rightarrow +\infty$ and $F \rightarrow \delta$, $F \rightarrow 1$. This permits the interpretation that it is the point structure of the cloud that actually causes the vanishing of the interaction. The resulting free (dressed!) physical particle – it is of course not identical with the bare one – behaves in virtue of its singular structure like an uncharged object that carries in itself the potentiality of evolving into a charged particle with the increase of the dimension of the cloud. This evolution is described by the structure function $F(x, g^2)$ or, equivalently, by $F(p, g^2)$. The dynamical form factors impart to the free uncharged physical particles at the same time both a charge capable of interaction and a probabilistic charge spread (or cloud structure) the latter being superimposed to the kinematical (mass-) spread the free particles possess in virtue of the combination of relativistic and quantal properties. Hence, there cannot be interaction (and/or charge) if there is no extended cloud (or charge spread) and vice versa. It is clear, therefore, that the F_n, F_n are the physically relevant quantities. As has been pointed out before, these form factors actually play the role of contributions to the exact vertex part, into which their effects will finally be transferred, and it is therefore not surprising that they are intimately related to quantities such as the familiar electromagnetic form factors (cf., section 4.2). In an expansion in powers of g^2 one obviously destroys and neglects the cloud and charge structure of the physical particles simultaneously with neglecting higher orders in g^2 (also if totally iterated graphs are accounted for) thereby eliminating at the same time interaction at all. Instead, a physically meaningful expansion must be one in powers of the charge structure part $g_0 F(x, g^2)$ of the approximate interaction HAMILTONIAN $H'[F] = g_0 F * H$, thereby taking account of the fact that (also in the exact theory) the interaction should come in via the spread rather than via the numerical value of the charge.

The point cloud obtained in the limit $g^2 \to +0$ (where $F \to \delta$) obviously is the smallest structure whose support has a non-negative measure $(r \geq +0)$ and the ghosts sometimes conjured up (instead of zero interaction) just arise from ignoring that the point cloud sets a limit to the procedure: An inversion of the sign of the COULOMB potential at small distances (as if a support of negative measure would appear) and a corresponding violation of the positive definiteness condition in HILBERT space are inevitable consequences.

It is important to notice that for $g^2 \to \infty$ the approximate propagators pass into free ones while the approximate Z-factors tend to one. This apparent "freezing-in" of the vacuum requires a reinterpretation since $F_n \to 0$. and $F_n \to 0$ and likewise $H'[F_n] \to 0$ as $g^2 \to \infty$, although the successive approximations may be expected to be reliable only for bounded values of g^2. The reinterpretation amounts to the introduction of an effective renormalized charge, $g_r(g^2)$, at each step of approximation, with g^2 playing now the role of an internal parameter which characterizes the cloud structure of the particles rather than a coupling. In the exact theory reaching $Z=1$ for some finite g^2 would seem to imply the intervention of bound states. These questions will be discussed when we compare our results with those of GELL-MANN and LOW [9]. At any case, the fact that $F_n \to 0$ for $g^2 \to \infty$ might lead to the conjecture that also the exact vertex function and the form factors defined by it vanish for large g^2 if, as it should be the case in an axiomatic theory, the interaction causes the vanishing of the vertex part and that of the form factors for infinite momentum transfer. In quantum electrodynamics, the non-locality certainly will result in a lack of gauge invariance. We do not consider this as being a serious defect of the approximation scheme as long as there exists the possibility that the sequence of theories converges to the exact one in which the generalized WARD identity is satisfied. More precisely, we take the extended cloud structure, charge spread and consistent renormalizability to be of higher importance then the preservation of a local continuity equation. On the other hand, after all what has been said before, it may be asking to much of an approximate theory to satisfy all the requirements of the exact local theory.

[9] M. GELL-MANN and F. E. LOW, Phys. Rev. 95, 1300 [1954].

2. Renormalization in Terms of Dispersion Relations

To account for a consistent dressing of the physical particles and to render the renormalization process mathematically meaningful, we define the local hermitian interaction HAMILTONIAN

$$H' = g_0 H(x) \tag{2.1}$$

by the limit of the nonlocal HAMILTONIAN $H'[\Delta]$:

$$H' = \lim_{\Delta \to \delta} H'[\Delta] = \lim_{\lambda \to 0} \Delta(x, \lambda) * H'(x). \tag{2.2}$$

Here, $\Delta(x, \lambda)$ is a real auxiliary cutoff function with $\Delta \to \delta(x)$ for $\lambda \to 0$ and the symbol $*$ denotes the convolution in coordinate space so that

$$\delta * H' = H'[\delta] = H'.$$

In relativistic theories where, for example,

$$H = \prod_n O_n(x),$$

$\Delta * H'$ means

$$g_0 \int_{-\infty}^{\infty} (\Pi\, \mathrm{d}x_n)\, \Delta(x - x_1, x - x_2, \ldots, \lambda)\, \Pi\, O_n(x_n)$$

whereas in the LEE model Δ acts on the Θ-particle. In subsection 2.1 we restrict the discussion to the V-particle propagator of the LEE model and to the totally iterated bubble approximation of a relativistic boson propagator postponing generalizations to subsection 2.2. The following notation will be used: In the relativistic case, M denotes the square of the renormalized boson mass, $p = k_0^2 - \vec{k}^2$ while the variable m in the various spectral representations has the dimension of the square of a reciprocal length (we use $\hbar = c = 1$). In the LEE model, M denotes the renormalized mass of the V particle, the variable m has the dimension of a reciprocal length and $p = E$. The constant $a > 0$ denotes the lower bound of the continuous part of the mass spectrum, $0 \leq M \leq a$, p generally is supposed to be a complex variable and the physical propagators, self-energies etc. are defined in the usual way as boundary values of the corresponding functions of

p (i. e.. for a real $p \geq a$, a vanishingly small positive imaginary part is to be added to p in the integrals defining propagators etc.).

2.1. Axiomatized Iterated Bubble Approximation

Let $K(p)$ be the divergent lowest order unrenormalized proper self-energy of the particle in the local theory with $K(M)=\infty$, $K'(M)=\infty$ (in the LEE model, K represents the exact expression, of course). Suppose Λ to be such that $K(p)$ is given by the limit of the corresponding finite unrenormalized self-energy $K_\lambda(p)$ of the cutoff theory (defined by $H'[\Lambda]$), viz. $K(p)=\lim_{\lambda\to 0} K_\lambda(p)$.

$K_\lambda(p)$ has the spectral representation

$$K_\lambda(p)=\int_a^\infty \mathrm{d}m\, g_0^2\, \varrho_\lambda(m)/(p-m) \tag{2.3}$$

with $\varrho_\lambda(m)=\varrho(m)\cdot \Lambda^2(m,\lambda)\geq 0$, (2.4)

the absorptive part of K being given by

$$\varrho(m)=\lim_{\lambda\to 0} \varrho_\lambda(m)\geq 0 .$$

Here. $\Lambda(m,\lambda)\geq 0$ is the cutoff function of the self-energy induced by the cutoff $\Lambda(x,\lambda)$ of the HAMILTONian $H'[\Lambda]$ with $\Lambda\to 1$ for $\lambda\to 0$. Generally. Λ is a functional of Λ,

$$\Lambda(x,\lambda)=T[\Lambda(m,\lambda);x]. \tag{2.5}$$

In the LEE model, Λ is essentially the FOURIER transform of $\Lambda[(k^2+\mu^2)^{1/2}+\varkappa,\lambda]$, $\varkappa$ and μ being the masses of N and Θ particle respectively and

$$a=\varkappa+\mu. \quad \varrho(m)=[(m-\varkappa)^2-\mu^2]^{1/2}.$$

In the case of a relativistic boson propagator, Λ acts on the two spinors in H' and is related with Λ through a double integral equation. In quantum electrodynamics and ps meson theory, $\varrho(m)$, a, and the unrenormalized coupling $g_0^2\geq 0$ are given by

$$\varrho=(m+2\,m_e^2)\,(1-4\,m_e^2/m)^{1/2},$$
$$a=4\,m_e^2. \qquad g_0^2=e_0^2/12\,\pi^2$$

and

$$\varrho=(m^2+4\,m\,m_n^2)^{1/2}\,\Theta(m-4\,m_n^2),$$
$$a=(3\,\mu)^2. \qquad g_0^2=g_{0\mathrm{ps}}^2/4\,\pi^2$$

respectively where. m_e, m_n and μ are the renormalized electron, nucleon and meson masses.

[10] Some properties of the cutoff theory are discussed in the Appendix.

By starting from the conventional HAMILTONian and S-matrix formalism, the unrenormalized boson or V-particle propagator in momentum space in the cutoff theory is given by

$$G_{u\lambda}(p)=[p-M-K_\lambda(p)+\delta M_\lambda]^{-1} \tag{2.6}$$

where the mass renormalization has been carried out according to

$$\delta M_\lambda=K_\lambda(M). \tag{2.6 a}$$

The cutoff function $\Lambda(m,\lambda)$ will therefore be chosen such as to make $K_\lambda(p)-\delta M_\lambda$ finite[10]. Hence, the unrenormalized propagator $(0\leq g_0^2<\infty)$

$$G_{u\lambda}(p)=G_0\Big/\left[1-g_0^2\int_a^\infty \mathrm{d}m\, \varrho_\lambda(m)/(p-m)\,(m-M)\right] \tag{2.7}$$

has no singularity in the complex p-plane cut from $p=a$ to $p=\infty$ except the pole at $p=M$ of the free propagator $G_0=1/(p-M)$ with residuum $Z_\lambda=1/(1-K_\lambda'(M))$; in particular. $G_{u\lambda}$ has no ghost pole. Therefore, by CAUCHY's theorem. with the path of integration taken along both sides of the cut and closing by an infinite circle, what gives the contribution

$$2\,i\,\mathrm{Im}\,G_{u\lambda}=-2\,i\,|G_{u\lambda}|^2\,\mathrm{Im}\,G_{u\lambda}^{-1}$$
$$=-2\,\pi\,i\,g_0^2\,\varrho_\lambda\,|G_{u\lambda}|^2,$$

the unrenormalized propagator satisfies the relation

$$G_{u\lambda}(p)=Z_\lambda G_0-\int_a^\infty \mathrm{d}m\, g_0^2\, \varrho_\lambda(m)\,|G_{u\lambda}(m)|^2/(p-m) \tag{2.8}$$

for all p and for all values $0\leq g_0^2<\infty$ of the unrenormalized coupling g_0^2.

From (2.8), the renormalized propagator.

$$G_\lambda(p)=Z_\lambda^{-1}G_{u\lambda}(p) \tag{2.9}$$
$$=G_0\Big/\left[1-(p-M)\,g^2\int_a^\infty \mathrm{d}m\, \varrho_\lambda(m)/(p-m)\,(m-M)^2\right]$$

is seen to satisfy the axiomatically correct relation

$$G_\lambda(p)=G_0+\int_a^\infty \mathrm{d}m\, g^2\, \varrho_\lambda(m)\,|G_\lambda(m)|^2/(p-m) \tag{2.10}$$

with the renormalized coupling $g^2=g_0^2 Z_\lambda$ [11] being now restricted to the interval

$$0\leq g^2<g_c^2(\lambda)\equiv\left[\int_a^\infty \mathrm{d}m\, \varrho_\lambda(m)/(m-M)^2\right]^{-1}. \tag{2.11}$$

[11] In the considered approximation, this relation will be used also in ps meson theory.

The rather trivial relations (2.8) and (2.10) may be considered as integral equations for or as integral (i. e., spectral) representations of the propagators. Equ. (2.10) has been derived by LSZ [12a, 12b] – in a less trivial way – from the general requirements of axiomatic field theory what justifies the fundamental role played by (2.10) in an attempt to axiomatize a HAMILTONian formalism. In fact, as will become clear in the following, extended cloud structure of physical particles, mathematically consistent renormalizability of the propagators for non-vanishing coupling and correct analytic structure of the propagators imply each other mutually.

If the restriction (2.11) is kept. then, since $g_c^2 \to 0$ in the local limit $\lambda \to 0$, (2.10) implies $G_\lambda \to G_0$. To escape this mapping of the theory into a free one [13], the original formalism must be extended by the inclusion of a continuation principle which releases the theory from the limitation imposed by (2.11), thereby providing a continuation to values $g^2 > g_c^2$ (to $g^2 > 0$ in the local limit) of the propagator defined by (2.9) for $g^2 < g_c^2$. The well-known ghost propagators $\hat{G}_\lambda(p)$ and $\hat{G}(p) = \lim_{\lambda \to 0} \hat{G}_\lambda(p)$ which cease to satisfy (2.10) for $g^2 > g_c^2$ are just the result of such an extension: $\hat{G}_\lambda(p)$ follows by continuation to values $g^2 > g_c^2$ of the propagator $G_\lambda(p)$ given by (2.9) and (2.10), i. e., of the *integral* on the right side of (2.10), by simply dropping the condition (2.11). This continuation obviously contradicts the positive-definiteness condition of the axiomatic theory and the correct charge-cloud structure of the particles [14]. It is precisely this continuation procedure which is implicitly performed in the conventional *purely formal* renormalization of the power-series representation of the S-matrix and in the formal perturbation theoretical solution of the LSZ equation system [15]. Actually, as we have seen, a mathematically and physically consistent renormalization of the local theory (within the frame of axiomatics) implies vanishing interaction in the approximation considered.

However, the necessity of an extension of the theory by a continuation principle being recognized, it is clear that the only continuation process that guarantees the axiomatic structure of the resulting propagator for all values $g^2 \geq 0$ consists of a continuation into the domain $g^2 > g_c^2$ of the *integrand of the integral representation* (2.10), i. e., of the spectral function $\sigma_\lambda(m, g^2)$ given for $g^2 < g_c^2$ by

$$\sigma_\lambda(m, g^2) = g^2 \varrho_\lambda(m) \, | G_\lambda(m) |^2 = [B_\lambda(m, g^2)/\pi(m-M)] \cdot [A_\lambda^2(m, g^2) + B_\lambda^2(m, g^2)] \quad (2.12)$$

where

$$A_\lambda(m, g^2) = 1 - (m-M) \, p \, v \cdot \int_a^\infty ds \, g^2 \varrho_\lambda(s) / [(m-s)(s-M)^2] \quad (2.13)$$

and

$$B_\lambda(m, g^2) = \pi \, g^2 \varrho_\lambda(m) / (m-M). \quad (2.14)$$

Then, the right side of (2.10) passes into a *definition* of an axiomatically correct propagator $G_{c\lambda}(p)$

[12a] H. LEHMANN, K. SYMANZIK, and W. ZIMMERMANN, Nuovo Cim. 2, 425 [1955]. By considering the quantity $g^2 \varrho_\lambda |G_\lambda|^2$ in the integrand of Eq. (2.10) as a given function, the r.h.s. of the equation is an integral representation for $G_\lambda(p)$ of the LEHMANN-KÄLLÉN type; cf. H. LEHMANN, Nuovo Cim. 11, 342 [1954]; G. KÄLLÉN, Helv. Phys. Acta 25, 417 [1952]. It is worth mentioning that the essential results of the present paper also remain valid if subtractions are included into the representation of the propagator.

[12b] It is perhaps not uninteresting to notice that there exists a very close analogy between the field theoretical formalism presented in this and in the next section and the formalism used for the description of the axiomatic theory of linear relaxation systems. Even the lowest-order spectral functions of both theories are almost the same. Cf. B. GROSS, Nuovo Cim. 3, Suppl. 2, 235 [1955]; B. GROSS, Theories of Viscoelasticity, Paris 1953; W. GÜTTINGER and B. GROSS, to be published.

[13] L. D. LANDAU, in „Niels Bohr and the Development of Physics“, edited by W. PAULI, London 1955.

[14] It should be clear that the appearance of a ghost-effect is *always* the result of a continuation procedure (although such continuation sometimes is made only tacitly). In the LEE model, a direct calculation of the eigen-states of the HAMILTONian shows that ghost states appear only if a continuation of the unrenormalized coupling constant to imaginary values is made whereas one arrives at zero interaction if one insists that the coupling be real and thereby renounces a continuation concept. That ghosts may arise in approximations to an exact axiomatic theory is due to the fact that the renormalization effects and the cloud structure of the physical particle are governed by those higher-order effects which the approximations do not yet include. In this case, the continuation (for a given $g^2 > 0$) which yields ghosts is the one in the variable x or p of the approximate propagators into domains for which the approximations cannot yet be thrusted since they do not include the renormalization effects that become important in those regions. Then, to avoid ghosts one has to modify the concept of the FOURIER transform operation by changing the integration measure into one depending on g. This is of course equivalent to introducing a coupling-dependent form factor. Thus, in any case, the entire difficulties seem to originate from the fact that an arbitrarily weak coupling, defined by an arbitrarily small g, may become arbitrarily strong the smaller the space-time region becomes in which the interaction takes place.

[15] H. LEHMANN, K. SYMANZIK, and W. ZIMMERMANN, Nuovo Cim. 1, 205 [1955].

which coincides with $G_\lambda(p)$ for $g^2 < g_c^2$:

$$G_{c\lambda}(p) = G_0 + \int_a^\infty \mathrm{d}m\, \sigma_{c\lambda}(m, g^2)/(p-m) \quad (2.15)$$

for all $g^2 \geq 0$ where the spectral function $\sigma_{c\lambda}(m, g^2) \geq 0$ is the (analytic) continuation in the variable g^2 of $\sigma_\lambda(m, g^2)$ to $g^2 > g_c^2$.

Let $p = p_\lambda(g^2)$ be the ghost pole, viz. the solution of the equation

$$G_0(p_\lambda)/\hat{G}_\lambda(p_\lambda) = 0. \quad (2.16)$$

The residuum of $\hat{G}_\lambda(p)$ at this pole is given by

$$N_\lambda(g^2) = \operatorname*{Res}_{p=p_\lambda}(\hat{G}_\lambda(p)) = g^2[\mathrm{d}(p_\lambda - M)/\mathrm{d}g^2]/(p_\lambda - M). \quad (2.17)$$

This is a particular case of Eq. (2.52). Since $G_{c\lambda}(p) = \hat{G}_\lambda(p)$ for $g^2 < g_c^2$, it follows that for $g^2 \geq 0$

$$G_{c\lambda}(p) = \hat{G}_\lambda(p) - N_\lambda(g^2)\, \Theta(g^2 - g_c^2)/(p - p_\lambda). \quad (2.18)$$

We shall prove in the Appendix that $G_{c\lambda}(p)$ has no zeros in the cut p-plane.

From (2.15) $G_{c\lambda}$ is seen to satisfy the relation

$$G_{c\lambda}(p) = G_0 - \int_a^\infty \mathrm{d}m\, g^2 \varrho_\lambda(m)\, F_\lambda^2(m, g^2)\, |G_{c\lambda}(m)|^2/(p-m) \quad (2.19)$$

for all $g^2 \geq 0$, $\lambda \geq 0$, with the g-dependent form factor

$$F_\lambda(m, g^2) = |\hat{G}_\lambda(m)/G_{c\lambda}(m)| = [1 + 2A_\lambda C_\lambda + C_\lambda^2(A_\lambda^2 + B_\lambda^2)]^{-1/2}. \quad (2.20)$$

Here, A_λ and B_λ are given by (2.13, 14) and C_λ is defined by

$$C_\lambda(m, g^2) = -N_\lambda\, \Theta(g^2 - g_c^2)(m - M)/(m - p_\lambda). \quad (2.21)$$

Since $G_{c\lambda}(p) \neq 0$ in the cut p-plane, it follows from (2.19) by a reversion of the argument similar to the one used in passing from (2.7) to (2.8) that $G_{c\lambda}(p)$ can be represented for all $g^2 \geq 0$ by the expression obtained from the right side of (2.9) by substituting there $g^2 \varrho_\lambda(m)\, F^2(m, g^2)$ for $g^2 \varrho_\lambda(m)$:

$$G_{c\lambda}(p) = G_0 \Big/ \Big[1 - (p - M) \int_a^\infty \mathrm{d}m\, g^2 \varrho_\lambda(m) \cdot F_\lambda^2(m, g^2)/(p-m)(m-M)^2\Big] \quad (2.22)$$

for $g^2 \geq 0$.

By comparing (2.22) with (2.9) and taking account of (2.2), (2.4) and (2.5), it is seen that $G_{c\lambda}(p)$ represents a propagator which could have been obtained by renormalizing – without encountering restrictions upon g^2 – a theory described by the non-local interaction HAMILTONian

$$H'[\Lambda * F_\lambda] = (\Lambda * F_\lambda) * g_0(g^2)\, H \quad (2.23)$$

with the particular g-dependent form factor

$$(\Lambda * F_\lambda)(x) = T[\Lambda(m, \lambda)\, F_\lambda(m, g^2); x] \quad (2.24)$$

[where $F_\lambda = F_\lambda(x, g^2)$] which is a uniquely determined functional of the product of the auxiliary cutoff Λ in the self-energy with the known function $F_\lambda(m, g^2)$ given by (2.20). $g_0(g^2)$ is defined by

$$g_0^2(g^2) = g^2 Z_{c\lambda}^{-1}(g^2) \quad (2.25)$$

where

$$Z_{c\lambda}^{-1}(g^2) = \lim_{p\to-\infty}(p - M)\, G_{c\lambda}(p) \quad (2.26)$$

for $g^2 > 0$ and $\lambda \geq 0$, and one finds

$$Z_{c\lambda}^{-1}(g^2) = (1 - g^2/g_c^2)^{-1} - N_\lambda(g^2)\, \Theta(g^2 - g_c^2) \quad (2.27)$$

$$= 1 + \int_a^\infty \mathrm{d}m\, g^2 \varrho_\lambda(m)\, F_\lambda^2(m, g^2)\, |G_{c\lambda}(m)|^2. \quad (2.28)$$

Obviously, we have

$$\left.\begin{aligned} F_\lambda(m, g^2) &= 1 \\ F_\lambda(x, g^2) &= \delta(x) \end{aligned}\right\} \quad \text{for} \quad g^2 < g_c^2(\lambda) \quad (2.29)$$

and the original cutoff HAMILTONian $H'[\Lambda]$ coincides with the new one, $H'[\Lambda * F_\lambda]$, for $g^2 < g_c^2$. The uniquely determined HAMILTONian $H'[\Lambda * F_\lambda]$ describes the "axiomatic continuation" into the domain $g^2 > g_c^2$ of the original theory which worked only for $g^2 < g_c^2$. As was to be expected, the superimposed form factor F_λ, which renders the theory to be axiomatic also for $g^2 > g_c^2$, depends explicitly on g. Of course, it would have been impossible to start with $\Lambda = \delta(x)$, i.e., with the local theory, since then the spectral function in (2.12) would be defined for $g = 0$ only and no continuation process whatsoever would give more than just the free theory. However, for carrying out the continuation and for constructing a unique axiomatized theory it is obviously sufficient that $\sigma_\lambda(m, g^2)$ be known for arbitrarily small values of g^2 and λ. Hence, the local H', furnished with an arbitrarily weak cutoff, actually is needed only in the limit of vanishingly small g^2 to determine the axiomatic theory (via the continuation of the spectral function) for larger values of g^2.

The behaviour of g_0^2 as a function of g^2 (Eq. (2.25)) is shown in Figs. 1 a and 1 b for $\lambda \neq 0$ and $\lambda \to 0$ respectively. These diagrams tell the whole story of the "axiomatization" of the HAMILTON formalism in the most obvious way. The second branch, $g^2 > g_c^2(\lambda)$, is in principle not attainable by the conventional HAMILTONian and S-matrix formalism alone but only by the inclusion of the continuation principle suggested by the general axioms of the theory. A detailed analysis of the $g_0^2 - g^2$-relation for more general propagators will be given in II. **Passing now to the limit $\lambda \to 0$, thereby eliminating** the auxiliary cutoffs Λ and $\bar{\Lambda}$ ($\Lambda \to \delta$, $\bar{\Lambda} \to 1$), we obtain the propagator of the "axiomatized iterated bubble approximation" for all values $g^2 > 0$:

$$G_c(p) = \lim_{\lambda \to 0} G_{c\lambda}(p)$$

$$= G_0 + \int_a^\infty \mathrm{d}m\, g^2 \varrho(m)\, |\hat{G}(m)|^2/(p-m)$$

$$= G_0 + \int_a^\infty \mathrm{d}m\, \sigma_c(m, g^2)/(p-m) \qquad (2.30)$$

$$= \hat{G}(p) + Z_c^{-1}/(p-p_0)$$

or

$$G_c(p) = G_0 \Big/ \Big[1 - (p-M) \int_a^\infty \mathrm{d}m\, g^2 \varrho(m) \cdot F^2(m, g^2)/[(p-m)(m-M)^2] \Big] \qquad (2.31)$$

where

$$\sigma_c(m, g^2) = [B(m, g^2)/\pi(m-M)] \cdot [A^2(m, g^2) + B^2(m, g^2)] \qquad (2.32)$$

and

$$F(m, g^2) = \lim_{\lambda \to 0} F_\lambda = |\hat{G}(m)/G_c(m)| = [1 + 2\,A\,C + C^2(A^2 + B^2)]^{-1/2}. \qquad (2.33)$$

Here,

$$A(m, g^2) = 1 - (m-M)\, p\, v \int_a^\infty \mathrm{d}s\, g^2 \varrho(s)/[(m-s)(s-M)^2], \qquad (2.34)$$

$$B(m, g^2) = \pi\, g^2 \varrho(m)/(m-M), \qquad (2.35)$$

$$C(m, g^2) = Z_c^{-1}(g^2)(m-M)/(m-p_0) \qquad (2.36)$$

are defined for $g^2 > 0$ and $p_0(g^2) = \lim_{\lambda \to 0} p_\lambda$ is the ghost pole, i. e., the solution of the equation

$$G_0(p_0)/\hat{G}(p_0) = 0\,.$$

The wave function renormalization is given by

$$Z_c^{-1}(g^2) = \lim_{p \to -\infty} (p-M)\, G_c(p) = 1 + \int_a^\infty \mathrm{d}m\, g^2 \varrho(m)\, |\hat{G}(m)|^2 \qquad (2.37)$$

or

$$Z_c^{-1}(g^2) = -g^2 (\mathrm{d}p_0/\mathrm{d}g^2)/(p_0 - M) = 1 + \int_a^\infty \mathrm{d}m\, \sigma_c(m, g^2) \qquad (2.38)$$

which is finite for $0 < g^2 < \infty$. Evidently, $Z_c = Z_2$ in the LEE model and $Z_c = Z_3$ in Quantum Electrodynamics or meson theory. G_c satisfies the relation:

$$G_c(p) = G_0 + \int_a^\infty \mathrm{d}m\, g^2 \varrho(m) \cdot \bar{F}^2(m, g^2)\, |G_c|^2/(p-m) \qquad (2.39)$$

$G_c(p)$ obviously is the renormalized propagator of a theory described by the non-local interaction HAMILTONian

$$H'[F] = F(x, g^2) * g_0(g^2)\, H(x) \qquad (2.40)$$

with the special form factor $F = \lim_{\lambda \to 0} \Lambda * F_\lambda = \delta * F$ being given by (2.23) with $\bar{\Lambda} \to 1$:

$$F(x, g^2) = T[\bar{F}(m, g^2); x] \qquad (2.41)$$

where $\bar{F}(m, g^2)$ is given by (2.33).

Thus, the axiomatization of the local HAMILTONian formalism results – to the order of approximation considered thus far – in a uniquely determined nonlocal interaction HAMILTONian, $H'[F]$, with the special g-dependent form factor $F(x, g^2)$. From (2.29) and (2.20) it follows that

$$\left.\begin{array}{l} F(x, g^2) \to \delta(x) \\ F(m, g^2) \to 1 \end{array}\right\} \text{asymptotically for } g^2 \to +0 \qquad (2.42)$$

and, therefore,

$$H'[F(x, g^2)] \to g_0'\, H(x) \quad \text{as } g^2 \to +0 \qquad (2.43)$$

where $g_0' \equiv g_0(0) = \lim_{g^2 \to 0} g_0(g^2)$ and

$$g_0^2(g^2) = g^2 Z_c^{-1}(g^2)\,. \qquad (2.44)$$

Hence, the original local interaction HAMILTONian $H' = g_0 H$ with the particular finite value $g_0 = g_0'$, which in general is different from zero, is asymptotic limit for $g^2 \to +0$ of the nonlocal interaction HAMILTONian $H'[F]$ [16].

2.2. *Generalizations*

The essential point to consider if one attempts to axiomatize DYSON's formal perturbation theoretical

[16] The belief that a nonlocal structure is hidden in the formalism set up by REDMOND and BOGOLIUBOV is already present in the final part of BOGOLIUBOV's paper [4]. After most of this work has been done last year, we have been informed by M. SCHÖNBERG that MEDVEDEV et al have also obtained a nonlocal HAMILTONian in the case of the LEE model by starting from BOGOLIUBOV's procedure.

renormalization scheme without being left with a free theory is that this scheme is mathematically consistent only if it is supplemented by a continuation principle that applies to the renormalized charge in the approximate propagators. In particular, a continuation of the absorptive part of the approximate self-energy gives rise to an apparently local theory with ghosts and requires the introduction of an indefinite metric in HILBERT space. On the other hand, the axiomatically correct continuation of the spectral function of the approximate propagator gives rise to a theory with a uniquely determined nonlocal HAMILTONian. A definite g-dependent form factor $F \neq \delta$ will always appear in $H'[F]$ if the exact vertex function of the original local theory does not vanish sufficiently rapidly for large momenta. This is the case, for example, in the LEE model where the above approximation already represents the exact expression. Then our approach yields a modified, however axiomatically correct theory. In local relativistic theories we do not know whether the interaction causes the exact vertex to vanish at large momenta. The conventional perturbation approximations to the vertex – via the self-energy – do not have this property but in fact give rise to the same situation as does the bubble approximation considered above. The spectral function $Q(m, g^2)$ of the exact renormalized proper boson (or V-particle) self-energy

$$W(p, g^2)/G_0 = (p-M)^2 \cdot \int_a^\infty \mathrm{d}m\, g^2\, Q(m, g^2)/[(p-m)(m-M)^2] \tag{2.45}$$

involves the vertex function according to [12]

$$Q(m, g^2) = \varrho(m)\, |\Gamma(m, g^2)/\gamma|^2 + U(m, g^2) \tag{2.46}$$

where Γ/γ is the ratio of the exact renormalized vertex function (for free particle momenta) to the zero-order vertex function, $U \geq 0$ is unknown and $\varrho(m)$ has been given in section 2 ($\Gamma = \gamma = 1$ in the LEE model where $U \equiv 0$, $\Gamma = \Gamma_5$, and $\gamma = \gamma_5$ in neutral ps meson theory). In analogy with Eqs. (2.10, 11) the exact renormalized propagator

$$G(p) = G_0/[1 - W(p, g^2)] \tag{2.47}$$

satisfies the relation

$$G(p) = G_0 + \int_a^\infty \mathrm{d}m\, g^2\, Q(m, g^2)\, |G(m)|^2/(p-m) \tag{2.48}$$

if

$$1 - Z(g^2) = I(g^2) = \int_a^\infty \mathrm{d}m\, g^2\, Q(m, g^2)/(m-M)^2 \leq 1\,. \tag{2.49}$$

$I = 1$ implies $\delta M = \infty$. The above relations also follow from the axiomatic formulation. From (2.39) and (2.46) it follows that $G_c(p)$ satisfies (2.48) if we put $\Gamma = \gamma F$ and $U = 0$. Although this choice is not unique, it still hints at the interpretation of F as a contribution to the exact vertex function. Of course, this problem should be further investigated in connection with higher-order perturbation approximations (cf., below); a rough estimation seems to confirm the guess but we have not analyzed the problem in detail. The spectral function $Q(m, g^2)$ in (2.45) is related to the spectral function $P(m, g_0^2)$ of the exact unrenormalized proper self-energy

$$\bar{K}(p) = \int_a^\infty \mathrm{d}m\, g_0^2\, P(m, g_0^2)/(p-m)$$

by $Q(m, g^2) = P[m, g^2 Z^{-1}(g^2)]$, $g^2 = Z(g^2)\, g_0^2$ and the exact unrenormalized propagator $G_u = Z G$ is given by $G_u = (p - M - \bar{K} + \delta M)^{-1}$ with $\delta M = \bar{K}(M)$.

The higher-order perturbation approximations $Q_n(m, g^2)$ (= polynomial in g^2) to $Q(m, g^2)$ do not decrease sufficiently rapidly for $m \to \infty$. Hence, introducing an auxiliary cutoff Λ_n according to $Q_n \to Q_n \Lambda_n^2$, the axiomatization of the n-th order approximate theory can be performed as that of the bubble approximation $Q_1 = \varrho_1 \equiv \varrho$. In particular, substituting $Q_n \Lambda_n^2$ for Q in (2.49) gives rise to a critical value $g_c^n(\lambda)$, and a continuation of the spectral function σ_λ^n of the approximate cutoff propagator beyond g_c^n gives, after passing to the limit $\lambda \to 0$, the axiomatized approximation to the exact propagator, G_c^n, with spectral function $\sigma_c^n(m, g^2)$. From the axiomatic formulation of field theory one knows that the spectral function in the propagator – and not the absorptive part of the self-energy – is the primary quantity that determines the dynamical properties of the system. As in the case of the axiomatized bubble approximation, the approximate propagator G_c^n contains a definite form factor $F_n(m, g^2)$ and belongs to a HAMILTONian $H_n' = H'[F_n]$ with form factor $F_n(x, g^2)$ where $F_n \to 1$ and $F_n \to \delta$ for $g^2 \to 0$. It is important to notice that in calculating, e. g., the next higher approximation to the self-energy, viz. $g^2 Q_2 = g^2 \varrho + g^4 \varrho_2(m)$, one could already use the axiomatized bubble approximation G_c rather than G_0 so that ϱ_2 actually becomes a functional of F, $\varrho_2(m) \to \varrho_2'[m, F(m, g^2)]$. This procedure – including also the correspondingly axiomatized fermion propagator – can in principle be iterated. However, except the axiomatized iterated bubble approximation G_c, probably none of the various modifications will lead to higher approximations G_c^n which are practically useful. Nevertheless, it is not impossible that a detailed analysis would give an indication whether the minimum of the function $g_0^2(g^2)$ increases or decreases if higher approximations are taken into account.

If the exact local theory exists, that is to say if the exact vertex function (more precisely, $g^2 Q$) vanishes for infinite momentum transfer, the sequence of form factors F_n must tend to $\delta(x)$ and the sequence of nonlocal interaction HAMILTONians $H'[F_n]$ consequently will tend to the local HAMILTONian $H' = H'[\delta]$. The question of convergence – and the g^2-domain for which it takes place – is of course as open as that of the behaviour of the exact vertex and at the present time we can only assert that $F_n \to \delta$ for $g^2 \to 0$ and for any finite n. At any case, it is clear that if the exact theory exists, then the appearance of the form factors in the approximate HAMILTONians is only a transitory aspect of the approximation scheme and the physical effects of the form factors F_n. F_n will finally be transferred into the exact vertex function or into the "internal" form factors defined by the latter. On the other hand, if the exact vertex part does not have the properties required by axiomatics, then our formalism supplies a definite nonlocal theory of which the local (but axiomatically inconsistent) theory may be an asymptotic representation for $g^2 \to +0$. In either case can the form factors $F \equiv F_1$ and F_n be viewed as playing the role of a vertex function or that of a contribution to it. That the form factors actually describe the cloud and charge structure of the physical particles has already been mentioned in the introduction. We shall come back to this question in sections 3 and 4.

Let now W_n/G_0 be the approximate (renormalized) self-energy of the boson which results from (2.45) by substituting there Q_n for Q. In calculating Q_n one may already use the "lower-order" axiomatized propagators $G_c^m(p)$ with $m \leq n-1$. Then the corresponding ghost propagator is obtained from (2.47) with W replaced by W_n:

$$\hat{G}^n(p) = G_0(p)/[1 - W_n(p, g^2)]. \qquad (2.50)$$

Under very general assumptions on W_n (they will be discussed in II) the axiomatized "n-th order" propagator takes (in analogy with (2.18) and (2.30)) the form

$$G_c^n(p) = \hat{G}^n(p) + Z_{cn}^{-1}/(p - p_{0n}) \qquad (2.51)$$

where Z_{cn}^{-1} is the corresponding axiomatized approximation to the exact Z^{-1} and $p_{0n}(g^2)$ is the ghost pole, i. e., the solution of the equation

$$G_0(p_{0n})/\hat{G}^n(p_{0n}) = 0. \qquad (2.52)$$

Z_{cn}^{-1} is given by

$$Z_{cn}^{-1} = (\mathrm{d}p_{0n}/\mathrm{d}g^2)/[(p_{0n} - M)\, \partial W_n(p_{0n}, g^2)/\partial g^2]$$
$$= -[(p_{0n} - M)\, \partial W_n(p_{0n}, g^2)/\partial p_{0n}]^{-1} \qquad (2.53)$$

in analogy with (2.38).

If, in particular, $W_n(p, g^2)$ is the n-th order perturbation theoretical expression, then

$$W_n(p, g^2) \simeq g^{2n} J(p)(p - M) \quad \text{for } g^2 \to +\infty \qquad (2.54)$$

and it follows that

$$Z_{cn}^{-1} \to 1 \quad \text{for} \quad g^2 \to +\infty. \qquad (2.55)$$

The implication of this result will be discussed in the next section. To prove (2.55), one observes that from (2.52) and (2.54) it follows that for $g^2 \to \infty$

$$p_{0n} = [g^{-2n}/J(M)] + M + o(g^{-2n}) \qquad (2.56)$$

whence

$$\partial W_n(p_{0n}, g^2)\,\partial p_{0n} = g^{2n}[J(M) + o(1)]$$
$$+ \mathrm{d}[\log J(M)]/\mathrm{d}M + o(1)$$

and, therefore,

$$Z_{cn}^{-1} = [\{[g^{-2n}/J(M)] + o(g^{-2n})\}\, \partial W_n/\partial p_{0n}]^{-1}$$
$$= 1 + o(1)$$

for $g^2 \to +\infty$. Here we have used the fact that, for $g^2 \to \infty$,

$$p_{0n}(g^2) \to M. \qquad (2.57)$$

3. Properties of the Theory

Most of the qualitative properties of the axiomatized theory have already been anticipated in the detailed discussion given in section 1. Referring to this section, we give in what follows a brief account of the relevant quantitative results that lend support to the interpretation given earlier. They follow directly by inserting the explicit expressions for $\varrho(m)$ into the propagators, form factors and normalization constants.

3.1. *Singular Structure and Asymptotic Properties*

In the limit $m \to \infty$, F behaves according to

$$F(m, g^2) \cong (Z_c^{-1} g^2 \log m)^{-1}. \qquad (3.1)$$

If g^2 is sufficiently small, $F \approx (g_0'^2 \log m)^{-1}$ for $g^2 > 0$, $m \to \infty$. In the limit of small g^2 ($g^2 \to +0$),

Z_c is given by

$$Z_c = (g^2/g_0'^2) \cdot \exp(-g_0'^2/g^2) \tag{3.2}$$

where

$$g_0' = 1 \tag{3.3}$$

in both the LEE model and the relativistic case (cf. also Ref. [3, 4]).

Since F and F_n tend to zero for $m \to \infty$, convergence is ensured for any finite order approximation if $g^2 > 0$. In the LEE model, for $\mu = 0$ and sufficiently small g^2 the equation for p_0 reduces to

$$[(\varkappa - p_0)/(M - p_0)] \cdot \log[(\varkappa - p_0)/(\varkappa - M)] = g^{-2} \tag{3.4}$$

whence $Z_c^{-1} = g^{-2}/[1 + (M - \varkappa)/(\varkappa - p_0)\, g^2]$ for $g^2 \to +0$, where $p_0 \to -\infty$. On the other hand, $p_0 \to M$ for $g^2 \to \infty$ in which case $Z_c \to 1$. In the relativistic case, the calculations are quite analoguous. $g_0{}^2$ increases monotonuously with increasing g^2 (see Fig. 1 b) and

$$Z_c^{-1} \simeq g_0'^2/g^2 \quad \text{for} \quad g^2 \to +0. \tag{3.5}$$

The charge of the cloud due to vacuum polarization is given by $g - g_0$ and, in virtue of the above results, the theory describes not bare particles for $g^2 \to +0$ but clothed physical particles whose bare core's charge ($g_0 \to g_0'$) becomes compensated by that of the cloud as $g^2 \to 0$. Hence the size of the cloud shrinks to the point occupied (or defined!) by the bare particle, the charge density of the cloud increases and finally "annihilates" that of the core, the result being free physical point-particles which do not interact with one another owing to their strong self-interaction. Since one cannot pass in a continuous way from $g^2 = 0$ to $g^2 > 0$, the free theory has to be defined as the limiting case $g^2 \to +0$ of the theory with interaction. This limit is of course a non-uniform one and can only be approached indefinitely. Although the interaction $H'[F]$ remains different from zero in this limit, according to (2.43), the particle is characterized by a free propagator: $G_c \to G_0$ for $g^2 \to +0$. It is readily verified that in virtue of the logarithmic behaviour of the renormalized self-energy not only has Z_c^{-1} an essential singularity at $g^2 = 0$ but so have also G_c, F and F and

$$G_c \simeq \hat{G},\ F \simeq \delta,\ F \simeq 1 \text{ asymptotically for } g^2 \to +0. \tag{3.6}$$

This also holds for any "finite-order" approximation if W_n increases logarithmically for $p \to \infty$ ($W_n \simeq (\log p)^r$, $\log\log p$ etc.) in which case it follows from (2.51) that

$$G_c{}^n \simeq G^n, \quad F_n \simeq \delta, \quad F_n \simeq 1$$
$$\text{asymptotically for } g^2 \to +0. \tag{3.7}$$

One therefore may ask more generally for the structure of axiomatic field theories the propagators of which have the ghost propagators of the conventional formally renormalized theories as asymptotic representations for $g^2 \to +0$. This gives of course a class of axiomatic theories in contrast with the unique theory obtained in this paper. One arrives at our unique $G_c{}^n$ if one requires that the class of axiomatic propagators $G_c{}^n$ which are asymptotically equivalent to $\hat{G}^n$ be restricted by the condition that disc $\{G_c{}^n\}$ = disc $\{\hat{G}^n\}$ on the cut or, what is the same, that the spectral function of the propagator be analytic in g^2. By starting from the requirement of asymptotic equivalence — in the sense of a physical correspondence principle — considerable information about the high-energy behaviour of field theories can be obtained and a mathematically correct meaning can be given to the method suggested by REDMOND, BOGOLIUBOV et al. for axiomatizing propagators. Of course, the physical meaning of such an approach — which has been discussed by one of us [5] — would remain obscure and could only be inferred (if at all) a posteriori, if one would not include form factors into the HAMILTONian which do not contribute to the formal series expansions in powers of g^2. At any case, the asymptotic properties of the theory make it evident that $\hat{G}^n$ and H' are needed only for $g^2 \to +0$ to determine the axiomatic theory. That $H' = \delta * H'$ furnishes the bare core only with a point-like cloud in a formal expansion of the S-matrix in povers of $g_0{}^2$ is immediately seen from the fact that Z_c^{-1} becomes infinite, in the way explained above, as $g^2 \to +0$ and in a power-series expansion, whereas Z_c^{-1} remains finite for $g^2 > 0$, in which case the form factor F in $H'[F] = F * H'$ is different from δ. The conventional perturbation approach in fact fails since it treats all distances in the same way irrespective of the magnitude of the coupling. The presented approximation scheme, on the other hand, ensures a correct prolongation of the cloud up to arbitrarily small distances, thereby treating each distance in a way compatible with the presence of interaction in the considered space-time region. As has been mentioned in section 1, a physically sensitive expansion has then to be the one in powers of $g_0 F$ where one may neglect higher orders (in $g_0 F$) without destroying the particles cloud or charge spread. It is obvious from the commutation relations

$$\langle [Z_c^{-1} \mathrm{d}\Phi(x,t)/\mathrm{d}t, \Phi(x',t)] \rangle_0 = \lim_{t\to t'} [\dot{G}_c(x_\nu, x_\nu')_{t>t'} - \dot{G}_c(x_\nu, x_\nu')_{t<t'}]$$

with $\dot{G}_c = \mathrm{d}G_c/\mathrm{d}t$ that the form factors F, F are causal form factors.

In particular, we have

$$\lim_{t\to 0} \lim_{g^2\to +0} [\dot{G}_c(x,t) - \dot{G}_c(x,-t)] = -i\,\delta(x) \quad (3.8)$$

and

$$\lim_{g^2\to +0} \lim_{t\to 0} [\dot{G}_c(x,t) - \dot{G}_c(x,-t)] = -i\,\delta(x) \cdot \lim_{g^2\to +0} (g_0'^2/g^2). \quad (3.9)$$

Eq. (3.8) tells us that two interacting physical particles behave as if they were free if one lets first shrink their clouds to the cores and then brings them together. Eq. (3.9) refers to the case where the interacting particles first are brought together and then their common cloud shrinks to a point. Clearly, the familiar statement that the renormalized equal-time commutators are at least as singular as the free one is now restricted to imply that they are less singular than the one corresponding to the limit of vanishing interaction. Eqs. (3.8), (3.9) are due to the fact that the limiting processes $g^2 \to 0$ and $p \to \infty$ do not commute with one another.

It is easily seen that mass renormalization is still infinite for bosons in the axiomatized iterated bubble approximation:

$$\delta M = Z_c \int_a^\infty \mathrm{d}m\,(m-M)\;\sigma_c(m,g^2) \quad (3.10)$$

($Z = Z_c = Z_3$) [17] reduces to

$$\delta M \simeq g_0^{-2} \int_a^\infty \mathrm{d}m/(\log m)^2 = \infty\,. \quad (3.11)$$

Generally, the exact boson mass renormalization will always be infinite in any axiomatic theory the boson propagator of which has a ghost-propagator as an asymptotic representation for $g^2 \to 0$. In particular, this holds for G_c^n. We show this in II. On the other hand, a generalization of the preceding formalism to relativistic fermion propagators shows that Z_2^{-1} as well as the mass renormalization for the fermions becomes finite in any axiomatic theory the fermion propagator of which has a ghost propagator as asymptotic representation for $g^2 \to +0$. This holds in particular for the axiomatized iterated bubble approximation of the fermion propagator — and likewise for higher approximations — of quantum electrodynamics and neutral ps meson theory. The reason for the fact that relativistic fermions' mass renormalization δM is finite in contrast with that of bosons (and with the δM of the V-particle) is the lower order degree of increase of the fermion spectral function (cf. II).

The approximation scheme we have suggested is equivalent to one in which one starts from a (local) HAMILTONian, applies a formal (re-)normalization to the physical states (cf. also section 4) obtained from the HAMILTONian and constructs from these states the renormalized axiomatic spectral function $\sigma(m,g^2)$ directly. For this purpose, H' is needed only in the asymptotic limit $g^2 \to 0$. While those approximations maintain the correct cloud structure, this structure is destroyed if one attempts to approximate formally the renormalized self-energy W or its absorptive part $g^2\,Q(m,g^2)$. The spectral function σ of the exact theory actually is the primary quantity in the axiomatic theory,

$$\sigma(m,g^2) = g^2\,Q(m,g^2) \cdot |\,G(m,g^2)\,|^2$$

[cf. (2.48)], while the absorptive part $g^2\,Q$ is the primary quantity in the HAMILTONian formalism.

3.2. Physical Meaning of g^2; Effective Charge

The renormalized charge g cannot directly be interpreted as being the true physical charge.

The relation (2.55), viz.

$$Z_c \to 1 \text{ and } Z_{cn} \to 1 \text{ for } g^2 \to \infty\,, \quad (3.12)$$

could lead one to believe that the vacuum is "freezing-in" since its "polarizability" $(g_0 - g)/g_0$ tends to zero for $g^2 \to \infty$ (although the successive approximations may be expected to be reliable only for bounded values of g^2; in the axiomatized LEE model we have a definite problem to solve since our first approximation already gives the exact theory). However, the correct interpretation of (3.12) rests upon the fact that, in the way it was obtained, the form factor $F(x,g^2)$ is not automatically normalized to unity except for $g^2 \to +0$. Indeed,

$$F \to 0, \quad F \to 0, \quad \int_{-\infty}^{+\infty} \mathrm{d}x\,F(x,g^2) \to 0$$

$$H'[F] = g_0\,F * H \to 0, \quad G_c \to G_0 \quad (3.13)$$

for $g^2 \to \infty$. This can directly be verified in the LEE model and by indirect arguments, starting from $F \to 0$, also in the relativistic case. We therefore have to replace F and g_0 in $H'[F] = g_0\,F * H$ by the effective form factor F_e and the effective un-

[17] For a finite mass renormalization, this axiomatic formula for δM can be shown to be equal to the one given by Eq. (2.6 a) respectively by the generalization of this equation (cf., II).

renormalized charge g_{0e}, respectively:

$$F_e(x, g^2) = F(x, g^2) \Big/ \Big[\int_{-\infty}^{+\infty} dx\, F(x, g^2) \Big], \quad (3.14)$$

$$g_{0e} = g_0 \cdot \int_{-\infty}^{+\infty} dx\, F(x, g^2). \quad (3.15)$$

Then the effective renormalized charge g_e, given by

$$g_e = g \int_{-\infty}^{+\infty} dx\, F(x, g^2), \quad (3.16)$$

is to be interpreted as being the true physical charge. Obviously,

$$g_e^2(g^2) = Z_c(g^2)\, g_{0e}^2(g^2), \quad (3.17)$$

together with $Z_c \to 1$ for $g^2 \to \infty$ implies that g_{0e}^2 is a two-valued function of g_e^2 with $g_e^2 \to 0$ for $g^2 \to \infty$ and $g_e^2 \approx g^2 \to 0$ for $g^2 \to 0$. g_{0e}^2 as function of g_e^2 is schematized in Fig. 2. It is clear then that g^2 does no longer play the role of the charge,

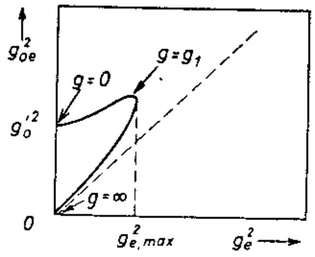

Fig. 2. The function $g_{0e}^2(g_e^2)$, parametrized by the variable g^2.

except for $g^2 \to 0$ where it coincides asymptotically with g_e^2. Instead, g^2 is now to be interpreted as being an internal parameter of the theory which characterizes the cloud structure of the physical particles or their interaction spread. Hence, while for $g^2 \to 0$ a free theory results in which the bare core is being compensated by the surrounding point cloud in the way described above with $H'[F] \to g_0' H \neq 0$, $g_e \to 0$, $g_{0e} \to g_0' \neq 0$, the theory passes into another free one for $g^2 \to \infty$ in the sense that $H'[F] \to 0$, $g_e \to 0$, $g_{0e} \to 0$, $G_c \to G_0$, where however the density of the cloud decreases with increasing cloud diameter. The function $g_e^2(g^2)$ is given in Fig. 3; its maximum gives that value g_1 of g up to which the approximation may be expected to be reliable. The same value g_1 also characterizes the maximal value that can be reached by g_e^2 considered as a function of g_{0e}^2. A glance at Fig. 2 shows that Gell-Mann and Low's assertion [9] $g_0 \equiv$ constant is confirmed in the limit of vanishingly small g_e, i.e., $d^n g_{0e}^2/d(g_e^2)^n \to 0$ for all $n \geq 1$ and $g_{0e}^2 \to g_0'^2$ as $g_e^2 \to 0$. This follows immediately from the corresponding expressions without the subscript "e", which are readily proved, taking into account that $F_e \to F$ for $g^2 \to 0$. However, the function $g_{0e}^2(g_e^2)$ always remains in the physical domain $Z^{-1} \geq 1$.

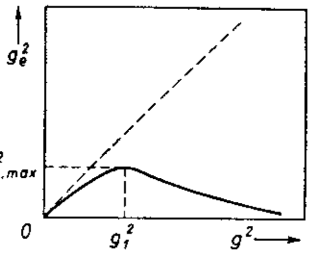

Fig. 3. The function $g_e^2(g^2)$.

From (2.55) we know that $Z = Z_{cn} \to 1$ for $g^2 \to \infty$ also if higher approximations are taken into account and that, therefore, $\bar{F}_n \to 0$, $F_n \to 0$ and $H_n' \to 0$ for $g^2 \to \infty$. Hence, the above reinterpretation of the theory in terms of an effective renormalized charge also applies to the higher axiomatized approximations and the function $g_{0e}^2(g_e^2)$ will always be of the type schematized in Fig. 2. If the approximate theories converge to the exact local one, then the essential characteristics of the function $g_{0e}^2(g_e^2)$ – namely that its diagram consists of a two-valued curve which starts at some point $g_e^2 = 0$, g_{0e}^2 and terminates at $g_e^2 = g_{0e}^2 = 0$ such that always $Z \leq 1$, – will be kept also for the exact theory. This implies, however, that Gell-Mann and Low's result, viz., $g_{0e}^2 = g_0'^2 =$ constant independent of g_e^2, can hold (if at all) only for $0 \leq g_e^2 \leq g_0'^2$ if $g_0'^2$ is finite and that the continuation of the function $g_{0e}^2(g_e^2)$, which is parametrized by the variable g^2, must consist either of the diagonal $Z = 1$ down from $g_{0e}^2 = g_0'^2$ to $g_{0e}^2 = 0$, $g_e^2 = 0$ or of any other curve inside the domain $Z < 1$ that reaches the origin with $Z = 1$. Hence, in contradistinction to Gell-Mann and Low's conjecture, there is no reason for excluding a finite $g_0'^2$.

On the other hand, while in the approximating curves $g_{0e}^2(g_e^2)$ the variable g^2 parametrizes the line and runs from zero to infinity (corresponding to $g_{0e}^2 = g_0'^2$, $g_e^2 = 0$ and $g_{0e}^2 = g_e^2 = 0$ respectively), it must be borne in mind that g_e^2 must become equal to g^2 if the exact theory is reached. Since, for a finite $g_0'^2$, g_e^2 necessarily is bounded by $g_0'^2$, whereas g^2 is not, g_e^2 cannot equal g^2 for $g^2 > g_0'^2$ except that either $g_0'^2 = \infty$ or the sequence converges not properly or something very particular will happen that makes g^2 to stop if the straight line $g_{0e}^2 = g_0'^2 =$ const. reaches the point characterized by $Z = 1$. We were not able to clarify this problem: it might be expected that reaching $Z=1$ for $g^2 = g_0'^2$ implies the intervention of bound states. Since the form factors effectively play the role of the vertex in the approximate theories, and vanish for $g^2 \to \infty$ one might conjecture then that also the exact vertex part vanishes for sufficiently large values of g if the interaction renders the vertex to vanish at infinite momentum transfer in a accordance with the requirements of an axiomatic theory. In this connection it is interesting to note that in virtue of $F \to \delta$, $F_n \to \delta$ for $g^2 \to 0$ and/or $n \to \infty$, the variables g^2 and $1/n$ play mathematically a role quite similar to the parameter ε in familiar representations of the delta function, $\delta(x) = \lim_{\varepsilon \to 0} \delta(x, \varepsilon)$. It might very well be that a detailed analysis of the peculiar property for $g^2 \to \infty$ of the approximating theories could give valuable informations about the structure of the exact local theory. For the time being, however, the results of this subsection merely imply that the approximate propagators may by physically useful for sufficiently small values of g^2 only. This and the fact that g^2 does no longer play the role of a coupling if it becomes to large, also prevents a discussion of strong coupling effects without further specifications.

3.3. *Potential between Charges* [18]

The finite structure of the axiomatized theory and the singular type of physical particle obtained in the limit $g^2 \to 0$ may of course most easily be visualized in terms of the potential between two charges. Taking quantum electrodynamics as basis, we obtain from the axiomatized iterated bubble approximation $G_c(p_\nu)$ the potential $U(r) = 3\pi V(r)$ with

$$V(r) = g^2 \int_{-\infty}^{+\infty} \mathrm{d}t\, G_c(x, t) \tag{3.18}$$

$$= g^2 \left(\frac{1}{r} - \frac{1}{r} \int_a^\infty \mathrm{d}m^2\, \sigma_c(m^2, g^2) \exp(-m\,r) \right)$$

$$= (g^2/r)\,[1 + h(r, g^2)]$$

with
$$h(r, g^2) = \int_a^\infty \mathrm{d}m^2\, \sigma_c(m^2, g^2) \exp(-m\,r) \tag{3.19}$$

where $G_c(x, t)$ is the FOURIER transform of $G_c(p_\nu)$ and $h(r, g^2)$ gives the measure for the deviation of the potential from the COULOMB one. Obviously the usual formally renormalized non-axiomatized iterated bubble approximation i.e., $\hat{G}$ could not have been used (it would change the sign of the potential at some $r = r_0$) while the non-iterated bubble approximation has been shown to be only a large distance approximation. From

$$\lim_{r \to 0} V(r) = g_0^2/r\,, \quad \lim_{r \to \infty} V = g^2/r$$

the usual interpretation of the deviation from the COULOMB potential as being due to the charge cloud around the bare core due to vacuum polarization follows. That is, the original point charge evolves into a extended object as the result of its interaction with the electron-positron-photon vacuum. From (3.19) and (2.37) it follows that

$$h(0, g^2) = Z_c^{-1} - 1$$

and
$$h(0, g^2) \simeq g_0'^2/g^2 \quad \text{for} \quad g^2 \to +0 \tag{3.20}$$

while for any finite $r > 0$ the convergence factor $\exp(-m\,r)$ in $h(r, g^2)$ and the fact that

$$\sigma_c(m, g^2) \simeq g^2 \varrho(m) \quad \text{for} \quad g^2 \to +0$$

implies that

$$h(r, g^2)_{r>0} \to 0 \quad \text{for} \quad g^2 \to +0\,. \tag{3.21}$$

Hence.
$$\lim_{r \to 0} \lim_{g \to 0} h(r, g^2) = 0 \tag{3.22}$$

and
$$\lim_{g \to 0} \lim_{r \to 0} h(r, g^2) = \infty\,. \tag{3.23}$$

This implies precisely that the charge of the core becomes completely compensated by that of the cloud in the limit $g \to 0$ where the size of the cloud shrinks to the point occupied by the bare core. Obviously (3.22) and (3.23) are equivalent to (3.8) and (3.9) respectively. For sufficiently small g^2 and $r \neq 0$ one has:

$$h(r, g^2) \simeq h_0(r, g^2) \tag{3.24}$$

$$= \int_a^\infty \mathrm{d}m^2\, g^2\, [\varrho(m^2)/m^4] \exp(-m\,r)\,.$$

The behaviour of h_0 for small and large r is given by

$$h_0 \simeq -2\, g^2 [\log(r\, m_e) + O(1)] \quad \text{for} \quad r\, m_e \to 0, \tag{3.25}$$

$$h_0 \simeq g^2 \pi^{1/2} (3/4) \exp(-2\, r\, m_e)/(r\, m_e)^{3/2} \tag{3.26}$$
$$\text{for} \quad r\, m_e \to \infty$$

respectively.

The substitution of h by h_0 can be considered a good approximation for $r \gg [\exp(-1/g^2)]/m_e$. However, the essential qualitative characteristics of h_0 for small r will be present in h also for $r \gtrsim [\exp(-1/g^2)]/m_e$.

With those few formulas one can indeed verify all the statements previously made about the cloud structure of the particles, such as the appearance of point structures and divergences ($Z_c^{-1} \to \infty$) for $g^2 \to 0$. It is likewise easily to be seen how the particle passes

[18] From here on, and also in section 4, we suppose m to have the dimension of a reciprocal length. In section 4 we shall use the notation $p = \mathfrak{p}$, $x = \mathfrak{r}$.

into a free uncharged one as $g^2 \to 0$ and similarly can the apparent "freezing-in" of the vacuum be inferred in the limit $g^2 \to \infty$ which has led us to the introduction of the effective charge g_e. The general behaviour of $h(r, g^2)$ for small g^2 is schematized in Fig. 4. The radius r_0 is defined by $h(r_0, g^2) \equiv \alpha \simeq 1$, whence

$$r_0 \simeq m_e^{-1} \exp(-1/g^2 3\pi). \tag{3.27}$$

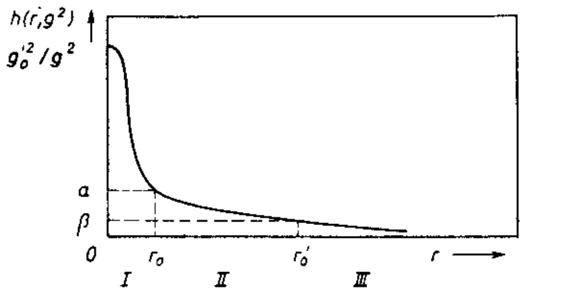

Fig. 4. The function $h(r, g^2)$, with $V(r) = g^2(1+h)/r$.

In the neighbourhood of r_0, h has a logarithmic behaviour and the region (I), $0 \leq r \lesssim r_0$, gives the main contribution to the renormalization effects. This domain is of course not attainable by a perturbation approach. r_0' is defined by $h(r_0', g^2) \equiv \beta$ with $\beta \simeq g^2$, whence

$$r_0' \simeq m_e^{-1}. \tag{3.28}$$

In the neighbourhood of r_0', h decreases exponentially, and r_0' is "stable" with respect to $g^2 \to 0$. It is clear that the domain (II), $r_0 \ll r < r_0'$, gives the relevant contributions to the radiative corrections to scattering and to the LAMB shift. In region (III), $r > r_0'$, the effect of the polarization part of the cloud can be neglected. As $g^2 \to 0$, h tends to zero in regions II and III while the size of the region I shrinks to zero with a simultaneous increase of h. If the effective charge g_e is introduced according to (3.16), h tends to zero everywhere as the structure parameter $g^2 \to \infty$ while the situation for $g^2 \to 0$ remains unaltered (since then $g \simeq g_e$).

One might expect that the picture would change completely if higher-order approximations are taken into account. However, it must be borne in mind that the situation encountered with at $g^2 \simeq 0$ is already the most singular case (in contrast with the perturbation theoretical result!). Therefore, and since $F \to \delta$ for $g^2 \to 0$, we are not so pessimistic to believe that the axiomatized iterated bubble approximation could not be thrusted at least for small g^2. We observe that r_0 could be identified with the gravitational radius $R = R_e$ of the electron if $3\pi g^2$ could be identified with the fine structure constant. However, in virtue of the exponential in (3.27), a small change in α produces a large change of r_0 even if α remains of the order one. This clearly implies that the familiar guess [13], viz., that the gravitational radius of the electron follows from the ghost pole $p_0(g^2)$ — with $G = D'_F$ —, is completely unjustified. In addition, we have seen that g does by no means play the role of the electron's charge. The only reasonable argument is the one referring to $r_0' = m_e^{-1}$ [Eq. (3.28)] which, by the choice $\beta \simeq g^2$, characterizes that domain [cf. (3.18)] $r \gtrsim m_e^{-1}$ in which only 4-th and higher orders of the charge play a role. One might perhaps speculate as to the situation in case that a "universal length" of the order $l = 10^{-13}$ cm should turn **out to play a role in physics. This seems suggestive, since the classical electron radius $r_0'' = e^2/m_e$ is of the same order as the radius $r_0''' = g_{ps}^2/m_n$ given by meson** theory. This situation, however, is not reproduced by the axiomatized theory, and the only length which is independent of the mass actually is given by the geometric mean $l_0 = (R \cdot r_0')^{1/2}$ where $R = m\varkappa$ is the gravitational radius of a particle with mass m and $\varkappa$ the gravitation-constant. In case of the electron, $m = m_e$ and **$l_0 \simeq 10^{-32}$ cm [19]. It is of course not impossible that the gravitational radius — actually the only critical length that so far appears to be incorporated in a consistent way into a physical theory — plays the fundamental role one expects from it, last but not least since R increases** with m while r_0' decreases with increasing m.

3.4. *Free Particle Spread*

We have thus far not explained whether or not the variables x (or r) and t can be considered as space-time coordinates. The arguments in section 3.3 apply, strictly speaking, only to fixed test charges but we consider the charge structure of the particles also as being associated with physical particles as a result of their interaction with the vacuum. The resulting charge spread is of course not to be confused with, but rather superimposed over, the mechanical (mass-)spread the non-interacting physical particles possess in consequence of the combination of relativistic and quantal properties (WEISSKOPF [20]). This spread is usually interpreted as being the result of virtual pair creation if a sufficiently accurate localization of the particles is attempted.

We are, however, not quite sure whether this interpretation is entirely consistent since the charge or mass distribution due to this spread does in no way depend on the coupling. That is to say, this spread of the free particles should rather be explained entirely within the frame of the free theory as being a purely kinematic effect. In fact, localization of a particle in a small domain prevents, in virtue of the uncertainty relation, any precise knowledge of its velocity and thus of its rest-system and, therefore, as a result of LORENTZ contraction, prevents any precise knowledge about the space extent of the particle in the measuring system. The argument is easily put into formulae, giving a

[19] D. I. BLOKHINTSEV, Fluctuation of Space-Time Metric, preprint, Dubna 1960.

[20] V. F. WEISSKOPF, Phys. Rev. **56**, 72 [1939].

spread of the order[21] $r_0' = m^{-1}$. One may associate this spread with the bare quantum and superimpose the (dynamical) spread of the cloud due to the interaction.

4. Cloud Structure and Interaction-Spread of Physical Particles

In this section we represent physical particle states in terms of bare ones to show that a divergent wave function (re-)normalization Z^{-1} due to an insufficiently decreasing vertex function (or, more generally, absorptive part of the self-energy) implies that the size of the cloud of virtual quanta in the physical states reduces to the point occupied (or defined) by the bare quantum (and vice versa). The infinite Z^{-1} that gives the point cloud and originates from an insufficient decrease of $g^2 Q(m, g^2)$ for $m \to \infty$, must of course be distinguished from the infinite Z^{-1} that would result in case g_0^2 were infinite and g^2 finite while $g^2 Q$ still decreases (as is the case, e. g., in the cutoff theory ($\lambda > 0$) for $g^2 \to g_c^2$) in which case the cloud still would have a finite extent. A finite Z^{-1} on the other hand corresponds to an extended cloud and conversely and only in this case is a consistent renormalization possible that leads to a theory of interacting particles. These statements hold of course for both exact and approximate theories. In the axiomatized theory described in the preceding sections, $Z^{-1}(g^2)$ is finite for $g^2 > 0$ but tends to infinity as $g^2 \to 0$ since then Q does no longer decrease sufficiently rapidly and the formal power series expansion of Z^{-1} has divergent coefficients. From this one can understand that the perturbation approximations reduce the cloud to a point. It should not be considered surprising then that the point cloud fails to reproduce any of the physical properties of the extended particle, in particular, that particles with a point-like cloud are not able to interact with one another (corresponding to a mapping of the theory into a free one). Of course, in renormalizing *formally* a field theory the above mentioned structures do not become explicit since such an approach does not tell anything about the cloud at small distances. We introduce, for the sake of generality, a parameter u such that $Z = Z_u = Z(u)$ becomes a function of u and that $Z_u^{-1} < \infty$ for $u > 0$ but

$$Z_u^{-1} \to \infty \quad \text{for} \quad u \to 0. \tag{4.1}$$

In accordance with what has been said before about the divergence of Z^{-1}, we suppose that Z_u^{-1} be representable by an integral which diverges for $u \to 0$, the integrand however remaining finite. In particular, u can represent the cutoff parameter λ if no axiomatic extension is performed. We shall, however, mostly interprete u as being the renormalized (effective) charge of our axiomatized theory, $u = g^2$ or $u = g_e^2$.

The fact that in describing the cloud or interaction spread of the physical particles we use a representation of physical states in terms of bare ones is of course subject to the familiar objection that the creation operators as well as the bare particle states are only formal entities that do not correspond to physically observable quantities. One might be tempted to avoid the bare particle concept at all by comparing the extended cloud of an interacting particle with the point-cloud defined by the free physical particle via a representation of the physical state of the former in terms of the states of the latter one. Since, however, the support of the singular free dressed particles obtained in the limit $g^2 \to +0$ coincides with the point defined by the bare particle one is left in the same boat with the previous problem. A mathematically more satisfactory approach might perhaps consist in using functional concepts. For instance, the size of the cloud of a particle cloud be defined by the support of a test function $\varphi(x)$. Then the point-like cloud will be described by the limit $\varphi \to \delta$ of the functional

$$[G(x), \varphi'(x)] = G[\varphi'] = \int \mathrm{d}x\, G(x)\, \varphi'(x),$$

i. e., by $\lim_{\varphi \to \delta} G[\varphi'] = \lim_{\varepsilon \to 0} [G(\varepsilon \cdot x), \varphi'(x)]$ which is essentially equivalent to the definition of $Z^{-1}\delta$.

It is clear, that the arguments of section 3.4 concerning the kinematical spread of the free particle also apply to the following discussion. It is mainly because of this spread that the (relativistic) creation operators do not create particles "at a point x", but in a domain with "center" at x.

4.1. *Non-relativistic Theory*

Let $|0\rangle$ denote the bare = physical vacuum and let $|p\rangle$ be a physical one-particle state (momentum $\vec{p} \equiv p$) with

$$P\,|p\rangle = p\,|p\rangle\,, \quad H\,|p\rangle = E\,|p\rangle \tag{4.2}$$

[21] Suppose the particle to have a spread Δl, i. e., that it could be localized in a region of dimension not smaller than Δl. The spread in the particle's rest-system would be $\Delta x = \Delta l/(1-v^2)^{1/2}$ ($\hbar = c = 1$). In virtue of the uncertainty relation $\Delta p\, \Delta l \geq 1$, the rest-system is subject to a uncertainty by an amount Δv, with $\Delta p = M\, \Delta v/(1-v^2)^{1/2}$. Hence, the minimal uncertainty is given by $\Delta x = 1/M$ since, in virtue of $(1-v^2)^{1/2} = [1 + (M\, \Delta l)^{-2}]^{1/2}$, we have $\Delta x = [(\Delta l)^2 + M^{-2}]^{1/2} \geq 1/M$.

where $H = H_0 + H'$ is the total HAMILTONian and let us denote by $a^+(p)$, $b^+(p)$, $c^+(p)$ the creation operators of a bare a-, b-, c-particle of momentum p. Then we have for the state of a physical a-particle, $|p, a\rangle$, the representation

$$|p, a\rangle = a^+(p)\,|0\rangle + \int dp'\, f_u(p, p')\, b^+(p-p')\, c^+(p')\,|0\rangle + \ldots \qquad (4.3)$$

the particular structure of which is given by the selection rules, f_u being a weight function. Introducing the states

$$|x, a\rangle = (2\pi)^{-3/2} \int dp\, |p, a\rangle \exp(-i p x)$$

and the operators

$$a^+(x) = (2\pi)^{-3/2} \int dp\, a^+(p) \exp(-i p x) \qquad (4.4)$$

etc. we have obviously

$$|x, a\rangle = a^+(x)\,|0\rangle - \int f_u(x-x', x'-x'')\, b^+(x')\, c^+(x'')\, dx'\, dx''\,|0\rangle + \ldots \qquad (4.5)$$

where

$$f_u(x, y) = (2\pi)^{-9/2} \int dp'\, dp''\, f_u(p', p'') \cdot \exp\{-i(p' x + p'' y)\}. \qquad (4.6)$$

The physical particle states thus far considered obviously are the unrenormalized states. We now have to renormalize (that is to say, to normalize) them. From (4.3) the normalization constant (= wave function renormalization) follows as being given by

$$1/\|\,|p, a\rangle\| = Z^{1/2}(u) \equiv Z_u^{1/2} = (1 + \int dp'\, |f_u(p, p')|^2 + \ldots)^{-1/2} \qquad (4.7)$$

and thus would seem to depend on p. If Z were independent of p, the (re-)normalized physical states will be defined by

$$A^+(p)\,|0\rangle = Z_u^{1/2}\,|p, a\rangle$$

with $\langle 0\, A^+(p')\, A^+(p)\, 0\rangle = \delta(p-p')$, (4.8)

$$A^+(x)\,|0\rangle = Z_u^{1/2}\,|x, a\rangle$$

with $\langle 0\, A^+(x')\, A^+(x)\, 0\rangle = \delta(x-x')$, (4.9)

where the scalar product of two states is defined according to

$$(A^+|0\rangle, A^+|0\rangle) \equiv \langle 0\, A^+\,|\, A^+\, 0\rangle \equiv \langle 0\,|\, A\, A^+\,|\,0\rangle. \qquad (4.10)$$

Hence, the renormalized physical creation operator is

$$A^+(x) = Z_u^{1/2}\, a^+(x) - \int dx'\, dx''\, s_u(x-x', x'-x'')\, b^+(x')\, c^+(x'') + \ldots \qquad (4.11)$$

where $\quad s_u(x, y) = Z_u^{1/2} f_u(x, y) \qquad (4.12)$

is a measure for the cloud structure or charge spread of the physical particle.

Let us now suppose that $f_u(p, p')$ in (4.3) is independent of p:

$$f_u(p, p') = f_u(p'). \qquad (4.13)$$

Physically this assumption obviously means that the (renormalized) masses m_a, m_b of the a- and b-particles are much larger than the mass m_c of the c-particle.

This also is clear since $f_u(p, p')$ behaves essentially like

$$[E_a(p) - \{E_b(p-p') + E_c(p')\}]^{-1} \simeq [p^2/2\,m_a - (p-p')^2/2\,m_b - p'^2/2\,m_c]^{-1} \to 2\,m_c/p'^2$$

if $m_a \gg m_c$, $m_b \gg m_c$. In consequence of this hypothesis, the wave function renormalization Z, Equ. (4,7), does no loger depend on p.

From (5.13) it follows that

$$f_u(x-x', x'-x'') = (2\pi)^3\, \delta(x-x')\, f_u(x-x'') \qquad (4.14)$$

where

$$f_u(x) = (2\pi)^{-3/2} \int dp\, f_u(p) \exp(-i x p). \qquad (4.15)$$

Hence

$$A^+(x) = Z_u^{1/2}\, a^+(x) + b^+(x) \int dx''\, Z_u^{1/2} f_u(x-x'')\, c^+(x'') + \ldots. \qquad (4.16)$$

The right side without further terms just reveals the situation given by the LEE model (a = V, b = N, c = Θ particle). Let us now interprete $A^+(x)$ as the operator which creates a physical particle whose "center is at x". The cloud's structure and size is defined through $s_u(x) = Z_u^{1/2} f_u(x)$. Confining ourselves to the LEE models by keeping only the first two terms on the right side of (4.16) — a discussion of more general schemes makes no difficulty — we have from (4.8, 9)

$$Z_u + \int dp'\, Z_u\, |f_u(p')|^2 = 1,$$

$$Z_u + \int dx\, Z_u\, |f_u(x)|^2 = 1. \qquad (4.17)$$

By hypothesis (4.1) we have in virtue of (4.7),

$$\lim_{u\to 0} \int dp'\, |f_u(p, p')|^2 = \infty \qquad (4.18)$$

and with (4.13)

$$\lim_{u\to 0} \int dp'\, |f_u(p')|^2 = \infty. \qquad (4.19)$$

This implies that under rather general assumptions about $f(p') = \lim_{u\to 0} f_u(p')$ the function $f(x) = \lim_{u\to 0} f_u(x)$ will be finite for $x \neq 0$. This is so, e. g., if

$$f(p') \simeq |p'|^\nu \quad \text{for} \quad p' \to \infty, \quad \text{with} \quad \nu \geq -3.$$

Since according to (4.1) $Z_u \to 0$ for $u \to 0$ it follows from (4.17) and (4.12) that for $s_u = Z_u^{1/2} f_u$ we have

$$\int dx\,|s_u(x)|^2 = \int dx\, Z_u\,|f_u(x)|^2 \to 1 \text{ for } u \to 0. \quad (4.20)$$

Hence, for $|s(x)|^2 = \lim_{u\to 0} |s_u(x)|^2$ it follows that

$$\int dx\,|s(x)|^2 = 1 \text{ and } |s(x)|^2 = 0 \text{ for } x \neq 0. \quad (4.21)$$

Therefore,
$$|s(x)|^2 = \delta(x). \quad (4.22)$$

This proves our statement: The probabilistic cloud density $|s_u(x)|^2$ reduces to a point-like quantity for $Z_u^{-1} \to \infty$. It is evident that the important point of the consideration is the dependence of the cloud's size on the magnitude of u. In our case $u = g^2$ is the renormalized charge, and the $f_u(x)$, $f_u(p)$ are readily expressed in terms of $F(x, g^2)$, $\tilde{F}(p, g^2)$ [cf. (4.45)].

If we would not have imposed the condition $f \simeq |p'|^{\nu}$ for $p \to \infty$, $\nu \geq -3$, we could of course have arrived at an $f(x)$ having a displaced singularity which would lead to a displaced point-cloud with density $\delta(x - x' - d)$ rather than $\delta(x - x')$ what would contradict rotational invariance. On the other hand, if

$$f(p') \to |p'|^{\nu} \exp(i\,d\,|p'|) \quad \text{for} \quad p' \to \infty$$

we would have a cloud on the surface of a sphere of radius d with center at the bare core. A field theory giving such a particle would contain "d" as an "elementary length". In addition one probably would arrive at instantaneous action at a distance and accausality, because of the peculiar structure of the cloud.

4.2. *Relativistic Theory*

We present the arguments in a rather schematic way for the relativistic case. The essential point to consider is now of course the non-equality of physical and bare vacuum.

The physical vacuum $|\tilde{0}\rangle$, defined by $P_\mu|0\rangle = 0$, can formally be expressed in terms of the bare vacuum $|0'\rangle$, defined by $P_{0\mu}|0'\rangle = 0$, according to (we write $f_u \equiv f$)

$$|0\rangle = |0'\rangle + \int dp\,dp'\,dp''\,f(p, p'\,p'')\,\delta(p + p' + p'') \cdot a^+(p)\,b^+(p')\,c^+(p'')\,|0'\rangle + \dots \quad (4.23)$$

where we have excluded interactions with derivative coupling. The norm

$$\langle 0|0\rangle = \exp[\delta(0)\,L] \quad (4.24)$$

is infinite: L is a divergent integral and $\delta(0)$ represents the infinite space volume. Of course, one can formally introduce a normalized physical vacuum,

$$|0\rangle = \exp[-\delta(0)\,L/2]\,|0\rangle \quad (4.25)$$

so that $\langle 0|0\rangle = 1$. In order to get a reasonable definition of cloud or interaction spread, however, it is essential to define the cloud structure in relation to the physical vacuum, i. e., in such a way that the term $\exp[-\delta(0)\,L/2]$ does not enter into its definition.

To this end, let us introduce the physical particle state $|p\rangle$, with $P_\mu|p\rangle = p_\mu|p\rangle$, $p_\nu^2 = m^2$, through a creation operator $A^+(p)$ such that the state of the (relativistic) a-particle is given by

$$\begin{aligned} |p, a\rangle &= A^+(p)|0\rangle, \quad A(p)|0\rangle = 0, \\ \langle p, a|p', a\rangle &= \delta(p - p'), \\ [A(p), A^+(p')]_\pm &= \delta(p - p'). \end{aligned} \quad (4.26)$$

Similarly for the relativistic b-, c-particles. A, A^+ may, for instance, be given through the Fourier representation of the ingoing fields:

$$A_{\text{in}}(x_\nu) = (2\pi)^{-3/2} \int dp[2\,\omega(p)]^{-1/2} \cdot [A^+(p)\exp(+i\,p_\nu x_\nu) + A(p)\exp(-i\,p_\nu x_\nu)]. \quad (4.27)$$

Of course, the A used here are of an entirely different nature as compared with the operators used in the non-relativistic case. The latter ones only involved creation operators, did not satisfy the commutation relations and described essentially the cloud of the physical one-particle state. On the other hand, the relativistic operators generally are functionals of bare creation and annihilation operators and describe, by iterated application to the vacuum state $|0\rangle$, both scattering and one-particle states.

The essential point is now to define the cloud or interaction spread in terms of physically "commensurable quantities". To this end we introduce a "bare with respect to physical vacuum" state, defined by

$$a^+(p)\,|0\rangle,\ a(p)\,|0\rangle \quad (\neq 0) \quad (4.28)$$

and express these states in terms of the A-states according to

$$a^+(p) = \tilde{a}(p)\,A^+(p) + \dots \quad (4.29)$$

$\tilde{a}$ being a c-number. We show that if the a^+, a are the Fourier coefficients of the field $A(x_\nu)$ at the time $t = 0$ which give the correct renormalized mass m, i. e.,

$$\mathrm{A}(x) = (2\pi)^{-3/2} \int dp[2\,w(p)]^{-1/2} \cdot [a^+(p)\,e^{-i\,p\,x} + a(p)\,e^{+i\,p\,x}] \quad (4.30)$$

$[w(p)=\sqrt{p^2+m^2}]$, then $\breve{\alpha}(p)$ is a constant independent of p:

$$\breve{\alpha}(p)=\text{constant}=\breve{\alpha}\,. \tag{4.31}$$

It will turn out that

$$\breve{\alpha}=Z^{1/2}\,. \tag{4.32}$$

If we would have made the expansion (4.30) in terms of a "wrong" mass m' by replacing

$$w(p)=\sqrt{p^2+m^2} \quad \text{by} \quad \sqrt{p^2+m'^2} \quad \text{with} \quad m'\neq m$$

we would have arrived at an $\breve{\alpha}$ depending on p. For the proof we observe that

$$U^{-1}A(p)\,U=[\sqrt{w(p')/w(p)}\,]\,A(p') \tag{4.33}$$

under the LORENTZ rotation $U=U(L)$ where $p_\nu'=L\,p_\nu$ and $U^{-1}A(x_\nu)\,U=A(x_\nu')$. Then, with the notation

$$\langle 0\,a^+\,|\,B^+\,0\rangle\equiv\langle 0\,|\,a\,B^+\,|\,0\rangle\equiv(a^+\,|\,0\rangle,B^+\,|\,0\rangle),$$

we have

$$\langle 0\,|\,A(x_\nu)\,A^+(p)\,|\,0\rangle = \langle 0\,|\,A(x_\nu')\,\sqrt{w(p')/w(p)}\,A^+(p')\,|\,0\rangle. \tag{4.34}$$

With $(x,t)=(x',t')=0$ we obtain from (4.33, 34), $\breve{\alpha}(p)=\breve{\alpha}(p')$, i.e., (4.31, 32). It is readily verified that the Z in (4.32) is identical with the one obtained from the propagator $G=\langle 0\,\,TA(x)\,A(x')\,\,0\rangle$ and that

$$Z^{1/2}\,\delta(p-p')=\langle 0\,a^+(p')\,|\,A^+(p)\,0\rangle\,. \tag{4.35}$$

Therefore, Z is *not* a measure for the probability of finding a bare in a physical state but gives a measure for the probability of finding a "bare wiht respect to the physical vacuum" – state in a physical state.

Let now f_1, f_2 be defined according to

$$\langle 0\,a^+(p)\,|\,B^+(p')\,C^+(p'')\,0\rangle = Z^{1/2}\,f_1(p',p'')\,\delta(p-p'-p''),$$

$$\langle 0\,a(p)\,|\,B^+(p')\,C^+(p'')\,0\rangle = Z^{1/2}\,f_2(p',p'')\,\delta(p+p'+p'') \tag{4.37}$$

and let us consider, in particular, quantum electrodynamics and take a-particle = photon, b = electron, c = positron (spin and polarization indizes are, of course, neglected). The state $B^+(p')\,C^+(p'')\,|\,0\rangle$ in (4.37) represents then an electron-positron scattering state. Let us introduce

$$a^+(x)=(2\pi)^{-3/2}\int dp\,a^+(p)\,e^{-ipx},$$

$$B^+(x)=(2\pi)^{-3/2}\int dp\,B^+(p)\,e^{-ipx}, \tag{4.38}$$

$$C^+(x)=(2\pi)^{-3/2}\int dp\,C^+(p)\,e^{-ipx}.$$

Then it follows that

$$\langle 0\,a^+(x)\,|\,B^+(x')\,C^+(x'')\,0\rangle = Z^{1/2}\,f_1(x-x',x-x'') \tag{4.39}$$

and

$$\langle 0\,a(x)\,|\,B^+(x')\,C^+(x'')\,0\rangle = Z^{1/2}\,f_2(x-x',x-x'') \tag{4.40}$$

where now of course $Z=Z_3$ and

$$f_{1,2}(x,y)=(2\pi)^{-9/2}\int f_{1,2}(p,p')\,e^{ipx+ip'y}\,.$$

We therefore may interprete $Z_3^{1/2}f_1$ and $Z_3^{1/2}f_2$ as being a measure for the "interaction-spread" of electron-positron scattering. And indirectly these quantities give a measure for the cloud structure. Indeed, $Z_3^{1/2}f_1(x-x',x-x'')$ gives the probability amplitude for finding a "bare with respect to physical vacuum"-photon at the point x in an electron-positron state with the "centers" of the electron and positron being at x' and x'', respectively. COMPTON scattering, via $\langle 0\,b^+\,|\,B^+\,A^+\,0\rangle$ and $Z=Z_2$ can be treated in the same way.

From

$$\langle 0\,a^+(p')\,|\,a^+(p)\,0\rangle - \langle 0\,a(p')\,|\,a(p)\,0\rangle = \delta(p'-p) \tag{4.41}$$

and expanding via a set of physical intermediate states it follows that

$$Z\,\delta(p'-p)+Z\int dp''\,|f_1(p-p'',p'')|^2\,\delta(p'-p)+\dots$$
$$-Z\int dp''\,|f_2(-p-p'',p'')|^2\,\delta(p'-p)=\delta(p'-p)$$

and therefore

$$Z\int dp\,dp'\,|f_1(p,p')|^2+\dots \tag{4.42 a}$$
$$-Z\int dp'\,dp\,|f_2(p,p')|^2=(2\pi)^3(1-Z)\,\delta(0).$$

Keeping only the first terms one obtains

$$Z\int dx\,dy\,(|f_1(x,y)|^2-|f_2(x,y)|^2)=(1-Z)\,\delta(0) \tag{4.42 b}$$

and if now $Z^{-1}\rightarrow\infty$ according to (4.1) – we have dropped the index u! – then, assuming f_1, f_2 to be finite for $x\neq 0$, $y\neq 0$ as in the non-relativistic case, we arrive at the result

$$Z\,|f_1|^2=c_1\,\delta(0)\,\delta(x)\,\delta(y),$$
$$Z\,|f_2|^2=c_2\,\delta(0)\,\delta(x)\,\delta(y) \tag{4.43}$$

with $\quad c_1-c_2=1 \quad c_1>0, \quad c_2>0$

where $\delta(0)$ originates from the normalization (4.41) and may of course be eliminated by the use of SCHWARTZ testing functions. The $\delta(x)\,\delta(y)$ term in (4.43) clearly corresponds to a point cloud, or more precisely, to the fact that the interaction spread reduces to a point as $Z^{-1} \to \infty$.

It is clear that this approach differs in a very fundamental way from the one of the non-relativistic case. It is by no means possible to render the analogy closer because the "bare with respect to the physical vacuum"-states do not constitue an orthogonal set. Nevertheless, the physical analogy of the relativistic case as discussed here with the situation encountered with in the discussion of the potential in section 3.3 becomes clear if one remembers that the spectral function σ_c which enters the potential is just given by the square of a sum over physical intermediate states of the type $B^- C^+ | 0\rangle$ as given in (4.37).

The structure functions

$$s_i = Z_3^{1/2} f_i, \quad i = 1, 2, \qquad (4.44)$$

are functionals of $F(x, g^2)$ or $\bar{F}(p, g^2)$. While in the non-relativistic case, and in particular in the LEE model where

$$s_u(x) = -(g_0 Z_u^{1/2}/4\pi^2) \qquad (4.45)$$
$$\cdot \int_{-\infty}^{\infty} \mathrm{d}k\, e^{-ikx} F(w + \varkappa, g^2)/(2w)^{1/2}(w + \varkappa - M)$$

with $u = g^2$, $w = (k^2 + \mu^2)^{1/2}$, we can describe s_u by the functional derivative $\delta A^-(x)/\delta c^+(x')$, no such simple characterization is possible in the relativistic case.

It is easy to establish the connexion between our functions s_1, s_2 and the familiar electromagnetic form factors $\tilde{F} = (D'_F/D_F)\,\Gamma = (G/G_0)\,\Gamma$ (resp., $\tilde{F} = (G_c/G_0)\,\Gamma$) when the a-particle is a photon: These form factors are given by [22]

$$\langle p_b' | j(0) | p_b'' \rangle \qquad (4.46)$$
$$= (p_b' - p_b'')^2_\mu D'_F (p'_{b\nu} - p''_{b\nu})\, \bar{v}_b\, \Gamma(p'_{b\nu}, p''_{b\nu})\, v_b ,$$

$$\langle 0 | j(0) | p_c'' p_b' \rangle \qquad (4.47)$$
$$= (p_b' + p_c'')^2_\mu D'_F (p'_{b\nu} + p''_{c\nu})\, \bar{v}_c\, \Gamma(p'_{b\nu}, -p''_{c\nu})\, v_b$$

where p_b', p_c'' are the momenta of ingoing b- and c-particles (for example, nucleon and antinucleon respectively) and D'_F corresponds to G or G_c.

Defining the current j by

$$\Box\, (Z_3^{-1/2} A) = e\, j \qquad (4.48)$$

(with $e = g$) it follows immediately from (4.47) that

$$[s_1(p_b', p_c'') + s_2(p_b', p_c'')]/(2\pi)^{3/2}\,(2w)^{1/2} \qquad (4.49)$$
$$= e\, Z_3^{1/2}\,(p_b' + p_c'')_\mu^{-2} \langle 0 | j(0) | p_c'' p_b' \rangle$$

where $w = w(p_b' + p_c'') = p_b' + p_c''$ and, therefore,

$$s_1(p_b', p_c'') + s_2(p_b', p_c'') = e\, Z_3^{1/2} (2\pi)^{3/2} (2w)^{1/2} D'_F$$
$$\cdot (p'_{b\nu} + p''_{c\nu})\, v_c\, \Gamma(p'_{b\nu}, -p''_{c\nu})\, v_b . \qquad (4.50)$$

5. Conclusion

In sections 2 and 3 we have shown that the form factors $\bar{F}$, F_n can be interpreted as contributions to the vertex function making use of the fact that according to eq. (2.46) the absorptive part $g^2 Q(m, g^2)$ of the self-energy is bounded below by $g^2 \varrho\, |\Gamma/\gamma|^2$. Since however the contribution $g^2 \varrho\, |\Gamma|^2 |G|^2$ of the two-fermion state to the spectral function

$$\sigma(m, g^2) = g^2 Q(m, g^2)\, |G|^2$$

of the boson propagator (2.48) only gives a lower bound, this does not mean that $\bar{F}$, F_n actually represent vertex parts except that U in eq. (2.46) is zero (as, for example, in the LEE model) [23]. That is to say, although the F_n always contribute to the exact vertex, they can be considered as vertex functions only within a prescribed order of approximation. The projection of "bare with respect to physical vacuum"-states onto physical particle states described in section 4 confirms the above interpretation of the form factors from a more general point of view: The explicit dependence of the structure functions $s_i^2 = Z_c f_i^2$ on F and $\bar{F}$ (resp., F_n, F_ν) and the fact that $|s_i|^2 \to \delta$ as $Z^{-1} \to \infty$ – what happens in our case for $g^2 \to 0$, where $F \to \delta$ – imply that the form factors actually give a measure for the size and cloud-structure (or charge-spread) of the interacting physical particles as compared with the point structure of the free uncharged particles [24]. This interpretation is lent additional support by the intimate connection of the s_i with the familiar electromagnetic fermion form factors $\tilde{F} = (G/G_0)\,\Gamma$ [resp. $(G_c/G_0)\,\Gamma$] given by eq. (4.47), i. e., $\tilde{F} = (D'_F/D_F)\,\Gamma$ in usual notation. Since $G/G_0 \to Z^{-1} = Z_3^{-1}$ for $p = k_\nu^2 \to \infty$, it is obvious that the

[22] P. FEDERBUSH, M. L. GOLDBERGER, and S. B. TREIMAN, Phys. Rev. **112**, 642 [1958].

[23] There remains, of course, an arbitrary phase $\exp(i\,q\,(x))$ if one puts $\Gamma = \gamma F$ as in section 2. This phase obviously does not contribute to the spectral function but may be used to render Γ analytic in the cut complex plane.

[24] Those particles, as shown in sections 2 and 3, are nevertheless described by a non-vanishing interaction HAMILTONian ($g_0' H \neq 0$).

electromagnetic form factors only vanish at infinite momentum transfer simultaneously with Γ if Z^{-1} is finite, representing point structures as do the s_i and F for $g^2 \to 0$. It is clear that the form factors s_i and F — and not Γ — are the physically relevant, measurable quantities upon which a theory must be based and that these quantities actually describe the spread caused by and associated with a non-zero charge relative to the point structure associated with a zero charge. In this sense our previous statement about the role of the bare particles is to be understood: The fact that the cloud of the physical particle shrinks to the point occupied by the bare quantum as $g^2 \to 0$ means that the operator of the bare quantum only defines the point to which the size of the physical particle would reduce if its charge would tend to zero. That is to say, at least the coordinate space variable introduced by the bare operators would remain if one attempts to express a physical particle state with $g^2 > 0$ in terms of physical states with $g^2 \to 0$, thereby avoiding the explicit use of the bare particle concept. One knows that owing to the intrinsic kinematical spread due to the combination of relativistic and quantal properties the bare operators do not create particles at a point in coordinate space. On the other hand, the physical meaning of the coordinate space and its relation to space-time is not quite clear since the actual description of the physical properties of particles — also that of the cloud structure and charge spread! — is based on energetical considerations. By starting from operators in momentum space we have defined the (flat) coordinate space via the usual Fourier operator, and thus have arrived at the above picture. We cannot be sure a priori whether this Fourier operator — and, therefore, the Lorentz group — does not merely represent an asymptotic concept valid for (and defining) large distances and so might not apply to the charge structure functions.

It is clear that the formalism discussed in the preceding sections may also be applied to more complicated systems, as for example to the ghost state problem in the nonlinear spinor theory and to four-fermion interactions. The intimate connection of the results so far reported with Feynman's operator and proper-time calculus, functional methods and the techniques of Volterra and Lappo–Danilewsky for solving equations like that for the S-matrix without having recourse to coupling constant expansions lies on hand. The relation of those theories with the asymptotic structures encountered in sections 2 and 3 is easily made explicit by representing the spectral function $\sigma(m, g^2)$ of the propagator by a Stieltjes integral with respect to the coupling,

$$\sigma(m, g^2) = \oint \mathrm{d}z \, r(m, z) / (g^2 z + 1)$$

so that the propagator admits of a double representation,

$$G(p) = G_0 + \iint \mathrm{d}m \, \mathrm{d}z \, r(m, z) / (p - m) \, (g^2 z + 1),$$

and applying the classical techniques of generating functions, moment problem etc. [25]. Such a representation seems to be essential if one attempts to account for the fact that one cannot, in general, treat problems referring to coupling and interaction strenghts in a way completely independent of the space-time region in which the interaction takes place.

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Appendix

1. In order to prove that the propagator $G_{c\lambda}(p)$ given by (2.15) admits of the representation (2.22), let $L(p)$ be defined such that

$$G_{c\lambda}(p) = [p - M - L(p)]^{-1}. \tag{A.1}$$

From (A.1) and (2.15) it follows that

$$L(p)/(p-M)^2 = \left[\int_a^\infty \mathrm{d}m \, \sigma_{c\lambda}(m, g^2)/(p-m)\right] \cdot \left[1 + (p-M) \int_a^\infty \mathrm{d}m \, \sigma_{c\lambda}(m, g^2)/(p-m)\right]^{-1}. \tag{A.2}$$

[25] We remark already at this occasion that the divergent contributions to the perturbation theoretical Z-faktor merely reflect the fact that by an expansion in powers of g^2 one actually represents the cloud structure function with extended carrier in terms of series of derivatives of delta-funktions with point-like carrier at the origin. Such an expansion of nonlocal quantities like the charge spread in terms of local ones generally gives rise to asymptotic expansions in momentum space. Cf., W. Güttinger, Singular Operations in Theoretical Physics, Sao Paulo 1957; Nuovo Cim. **10**, 1 [1958]; Nucl. Phys. **9**, 429 [1959].

mum of $g_0^2(g^2)$, i. e., $g'^2_{0\lambda}$, in the second branch ($g^2 > g_c^2$) tends to $g_0'^2$ as $\lambda \to 0$ in which limit the entire part of Fig. 1 a situated to the left of the minimum degenerates into the straight line ($g_0^2 > 0$, $g^2 = 0$).

3. The difficulties one would encounter by using an arbitrary auxiliary cutoff may easily be inferred by considering a cutoff of the type $\bar{\Lambda}^2(m, \lambda) = \Theta(\lambda^{-1} - m)$ where $1/\lambda > a$ [with $\Theta(x) = 1$ or $= 0$ for $x > 0$ or $x < 0$, respectively]. In this case, $G_\lambda(p)$ will have an additional pole for some $p > 1/\lambda$ if g^2 lies in the interval $\bar{g}^2 \leq g^2 < g_c^2$ where $\bar{g}^2$ is given by Eq. (A.14) below. Let $G_\lambda(p)$ be given by (2.9) and consider the first-order renormalized self-energy $W_{1\lambda}(p)/G_0$ in the cutoff theory with

$$W_{1\lambda}(p) = (p - M) \int_a^\infty \mathrm{d}m\, \varrho_\lambda(m)/[(p-m)(m-M)^2]\,, \tag{A.11}$$

so that $G_\lambda = G_0/(1 - W_{1\lambda})$. $W_{1\lambda}(p)$ is analytic in the cut p-plane and is real and monotonuous for $-\infty < p < a$ with

$$W_{1\lambda} \to g^2 \int_a^\infty \mathrm{d}m\, \varrho_\lambda(m)/(m-M)^2 \equiv g^2/g_c^2$$

for $p \to -\infty$. Suppose that $g^2 < g_c^2$. Since $W_{1\lambda} < 1$ for $-\infty < p \leq M$, G_λ has no ghost poles nor are there poles for $M < p < a$ where $W_{1\lambda} < 0$. On the cut, i. e., for $a < p < \infty$, we have in the limit $\varepsilon \to 0$,

$$W_{1\lambda}(p \pm i\varepsilon) = (p - M)\, p\, v \cdot \int_a^\infty \mathrm{d}m\, g^2 \varrho_\lambda(m)/[(p-m)(m-M)^2] \mp i\pi g^2 \varrho_\lambda(p)/(p-M) \tag{A.12}$$

and if $\varrho_\lambda(m)$ does not vanish for $a < m < \infty$, the equation $W_{1\lambda}(p) = 1$ has no solution for $a < p < \infty$, $G_\lambda(p)$ having therefore no additional poles. If, however, ϱ_λ has zeros then the additional poles will appear. In particular, with the above cutoff,

$$\varrho_\lambda(m) = \varrho(m)\ \Theta(\lambda^{-1} - m)$$

vanishes for $m > 1/\lambda$. Then, for $p > 1/\lambda$ the equation $W_{1\lambda}(p) = 1$ can be written as

$$g^{-2} - g_c^{-2} = \int_a^{1/\lambda} \mathrm{d}m\, \varrho(m)/[(p-m)(m-M)] \tag{A.13}$$

and since $\varrho(m) \geq 0$ the r.h.s. of (A.13) is non-negative and decreases from its value at $p = 1/\lambda$ to zero as p goes from $1/\lambda$ to infinite. Hence, in virtue of $g^2 < g_c^2$, the equ. (A.13) and therefore the equation $W_{1\lambda}(p) = 1$ will always have a solution for all g^2 in the interval $\bar{g}^2 < g^2 < g_c^2$ where $\bar{g}^2$ is given by

$$\bar{g}^2 = \left[g_c^{-2} + \int_a^{1/\lambda} \mathrm{d}m\, \varrho(m)/[(\lambda^{-1} - m)(m-M)]\right]^{-1}. \tag{A.14}$$

Consequently, $G_\lambda(p)$ has an additional pole on the cut side. The similar situation could arise from an auxiliary cutoff which makes ϱ_λ and, therefore, $\sigma_{c\lambda}$ to vanish at only a discrete set of points what eventually could introduce zeros of $G_{c\lambda}$ on the cut, thereby invalidating the arguments of Appendix 1. Since, however, neither the absorptive part $\varrho(m)$ of the self-energy of the original local theory nor the spectral function σ_c of the axiomatized propagator contains zeros, there is no reason for making the situation more complicated by using an auxiliary cutoff – which in addition is only of transitory nature – that gives rise to such additional zeros. Those auxiliary cutoffs will therefore be excluded from our considerations. Obviously, the zeros of the propagators that could appear on the cut are the familiar CDD-zeros [26].

[26] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 [1956].